Banach J. Math. Anal. (2023) 17:14
https://doi.org/10.1007/s43037-022-00238-x

Tusi Mathematical Research Group

ORIGINAL PAPER

Solution of an infinite band matrix equation

S. H. Kulkarni[1] · R. Radha[2] · K. Sarvesh[2]

Received: 14 August 2022 / Accepted: 21 November 2022 / Published online: 17 December 2022

Abstract
The aim of this paper is to obtain the solution of the operator equation $Tx = y$ for an infinite band matrix T using its finite-dimensional truncations $T_n x^n = y_n$. Several verifiable conditions are given to obtain the invertibility of T. An application in connection with a stable set of sampling for functions belonging to a shift-invariant space is discussed along with an illustration.

Keywords Band matrices · Finite dimensional truncations · Operator equations · Tridiagonal operator

Mathematics Subject Classification 65J10 · 47B37 · 65F50

1 Introduction

In 2006, Balasubramanian et al. [10] studied the solution of a tridiagonal operator equation $Tx = y$ on $\ell^2(\mathbb{N})$ using its finite sections $T_n x^n = y^n$. They showed that if $\left\{\left\|T_n^{-1}e_n\right\|\right\}$ and $\left\{\|T_n^{*^{-1}}e_n\|\right\}$ are bounded, then T is invertible and the solution can be obtained as a limit in the norm topology of the solutions of its finite sections. The main aim of this paper is to obtain a similar result for an infinite band matrix, so generalizing the findings in [10]. In numerical analysis, solutions of band matrix equations play a crucial role when one has to obtain numerical solution of ordinary

Communicated by Ngai-Ching Wong.

✉ R. Radha
radharam@iitm.ac.in

S. H. Kulkarni
shk@iitpkd.ac.in

K. Sarvesh
sarvesh.kmc4@gmail.com

1 Department of Mathematics, Indian Institute of Technology Palakkad, Palakkad 678557, India

2 Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India

and partial differential equations in connection with boundary value problems and numerical approximation methods of local type. These matrices arise while looking at the approximate solutions of such boundary value problems by local approximation methods such as finite differences, finite elements, isogeometric analysis etc. Some of the references in this direction are [9, 13, 17, 21, 31, 32]. Further regarding the invertibility of an operator, it is well known that if $T \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space, has a strict row and column dominance property then T is invertible. Further, an infinite version of Gerschgorin theorem gives the information about the spectrum of T. In fact, let $T = (\alpha_{ij})$, $r_i = \sum_{j \neq i} |\alpha_{ij}|$, $r_i^{'} = \sum_{k \neq i} |\alpha_{ki}|$. If $D_i = B(\alpha_{ii}, r_i)$ and $D_i^{'} = B(\alpha_{ii}, r_i^{'})$, then $Sp(T) \subset \overline{\left(\bigcup_i D_i \cup D_i^{'}\right)}$.

The method used in [10] makes use of determinants of T_n and their recurrence relations. In fact, the expressions for $T_n^{-1} e_n$ and $T_n^{*^{-1}} e_n$ were written in terms of determinants of $T_n, T_{n-1}, \cdots$ and T_1. This was a strong restriction as it dealt with the specific nature of tridiagonal structure and hence the result could not be generalized to infinite band matrices with bandwidth $2M + 1$ for an arbitrary M. Our new approach totally avoids the expressions in terms of determinants. In fact, we obtain the expressions for $T_n^{-1} e_{n-j}$ and $T_n^{*^{-1}} e_{n-j}$, $0 \leq j \leq M - 1$ using matrix equations. Before we mention about the contents of the paper, we shall give a brief introduction to the finite section method. We refer to [11, 16, 24, 30] in this connection.

Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let $\mathcal{H}_n$ denote the span of $\{e_1, \ldots, e_n\}$. Let $T \in \mathcal{B}(\mathcal{H})$, the class of bounded linear operators on $\mathcal{H}$. Let T_n denote $T_n = P_n T_{|\mathcal{H}_n}$ where P_n denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$. The $n \times n$ matrix T_n is called a finite section or a Galerkin approximation of T. In general in order to study the analytic properties of T one can first study the analytic properties of T_n and extend to the whole of T. This method is called finite section method or Galerkin method. The finite section method has been used in several contexts such as variational problems [18, 22], solutions of operator equations involving convolution operators, Toeplitz operators and block Toeplitz operators [11, 12, 23]. Another important problem is the study of spectrum of a self adjoint operator T on a Hilbert space using its finite sections. Arveson [7, 8] showed that only if the given operator T is viewed as an element of an appropriate C^* algebra, one can see the precise nature of limit of the eigenvalue distributions: the limit is associated with a tracial state on T.

One of the important aims of the paper [10] was to obtain sufficient conditions using the entries of T explicitly in order to show the boundedness of $\{\left\|T_n^{-1} e_n\right\|\}$ and $\{\|T_n^{*^{-1}} e_n\|\}$. A complicated theorem (Theorem 6.1 in [10] was proved and several sufficient conditions were obtained as corollaries. Recently in [6] Antony Selvan and Radha extended the study to a tridiagonal operator T on $\ell^2(\mathbb{Z})$ in obtaining the solution of $Tx = y$ using finite sections. They also proved that if T is a tridiagonal operator on $\ell^2(\mathbb{Z})$ which, is strictly row and column dominant except for a finite number of rows and columns, then T is invertible.

The aim of this paper is to study the invertibility of infinite band matrices T and solutions of their operator equations when $T \in \mathcal{B}(\ell^2(\mathbb{N}))$ and $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$ as well. We also wish to obtain the solution of such operator equations $Tx = y$ using its finite-dimensional truncations $T_n x^n = y_n$. Furthermore in the case of a tridiagonal operator, we give a verifiable condition to get the boundedness of $\{\left\|T_n^{-1}e_n\right\|\}$ and $\{\|T_n^{*^{-1}}e_n\|\}$ without using the complicated theorem (Theorem 6.1 in [10]) for the invertibility of T. In particular, we prove that if the product of consecutive diagonal elements in absolute value, $|d_n d_{n+1}|$, is large enough, then we can obtain the invertibility of T. In other words, even if infinitely many rows and columns lack diagonal dominance condition, we can establish the invertibility of T. We also extend the verifiability criterion for a doubly infinite tridiagonal operator. We illustrate the verifiability conditions by simple numerical examples. Furthermore, we extend the verifiability conditions for doubly infinite pentadiagonal operators which clearly show that the theory can be extended to a general $(2M+1)$ diagonal operator.

In the final part of the paper, an application in connection with a stable set of sampling for functions belonging to a shift-invariant space is discussed along with an illustration.

2 Matrices with bandwidth 2M+1 on $\ell^2(\mathbb{N})$

Let $\mathcal{H}$ denote a separable Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let $\mathcal{H}_n$ denote the linear span of $\{e_1, \ldots, e_n\}$ and P_n, the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Let $T : \mathcal{H} \to \mathcal{H}$ be a $2M+1$ diagonal operator defined as

$$Te_n = \begin{cases} \sum\limits_{j=1}^{n-1} u_{n-j}^j e_{n-j} + d_n e_n + \sum\limits_{j=1}^{M} l_{n+j}^j e_{n+j}, & n \le M, \\ \sum\limits_{j=1}^{M} u_{n-j}^j e_{n-j} + d_n e_n + \sum\limits_{j=1}^{M} l_{n+j}^j e_{n+j}, & n > M, \end{cases} \tag{2.1}$$

where $\{d_n\}$, $\{l_n^j\}$ and $\{u_n^j\}$ for $j = 1, \ldots, M$ are bounded sequences of complex numbers. Further, we assume that the sequence $\{d_n\}$ is bounded from below by a number $k_0 > 0$. Then $T_n : \mathcal{H}_n \to \mathcal{H}_n$ can be defined as

$$T_n = P_n T P_n = P_n T_{|\mathcal{H}_n}.$$

It is clear that matrix of T_n with respect to the orthonormal basis $\{e_1, \ldots, e_n\}$ consists of first n rows and n colums of T. The matrices T_n are known as finite sections or Galerkin approximations of T. In matrix notation T_n (for large n) can be written as

$$T_n = \begin{bmatrix} d_1 & u_1^1 & u_1^2 & \cdots & u_1^M & 0 & 0 & \cdots & 0 \\ l_2^1 & d_2 & u_2^1 & \cdots & u_2^{M-1} & u_2^M & 0 & \cdots & 0 \\ l_3^2 & l_3^1 & d_3 & u_3^1 & \cdots & u_3^{M-1} & u_3^M & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & l_{n-1}^M & l_{n-1}^{M-1} & \cdots & l_{n-1}^1 & d_{n-1} & u_{n-1}^1 \\ 0 & \cdots & 0 & 0 & l_n^M & \cdots & l_n^2 & l_n^1 & d_n \end{bmatrix}.$$

We can also write T_n(for large n) as

$$T_n = \begin{bmatrix} & & 0 \\ & & \vdots \\ & T_{n-1} & u^M_{n-M} \\ & & \vdots \\ & & u^1_{n-1} \\ 0 \cdots & l^M_n \quad \cdots l^1_n & d_n \end{bmatrix} = \begin{bmatrix} T_{n-1} & U_n \\ L_n & d_n \end{bmatrix},$$

where $U_n = (0, \ldots, 0, u^M_{n-M}, \ldots, u^1_{n-1})^T$, $L_n = (0, \ldots, 0, l^M_n, \ldots, l^1_n)$.

We assume that each T_n is invertible. Our first aim is to calculate $T_n^{-1} e_n$ and $T_n^{*^{-1}} e_n$. Let $T_n \bar{\mathbf{x}} = e_n$, where $\bar{\mathbf{x}} = (x_1, \ldots, x_{n-1}, x_n)^T$. We can write $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_{n-1}, x_n)^T$, $\bar{\mathbf{x}}_{n-1} = (x_1, \ldots, x_{n-1})^T$, 0_{n-1} is the $n-1$ dimensional vector of zeros. With the above notation we can write $T_n \bar{\mathbf{x}} = e_n$ as

$$\begin{bmatrix} T_{n-1} & U_n \\ L_n & d_n \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix},$$

which reduces to the system of equations

$$T_{n-1} \bar{\mathbf{x}}_{n-1} + U_n x_n = 0_{n-1} \tag{2.2}$$

$$L_n \bar{\mathbf{x}}_{n-1} + d_n x_n = 1. \tag{2.3}$$

Since T_n is invertible for each n, from (2.2) we obtain $\bar{\mathbf{x}}_{n-1} = -T_{n-1}^{-1} U_n x_n$. After substituting this value of $\bar{\mathbf{x}}_{n-1}$ in (2.3) we get

$$x_n = \frac{1}{d_n - L_n T_{n-1}^{-1} U_n} \quad \text{and} \quad \bar{\mathbf{x}}_{n-1} = \frac{-T_{n-1}^{-1} U_n}{d_n - L_n T_{n-1}^{-1} U_n}.$$

Let $T_{n-1}^{-1} U_n = a_1 e_1 + \cdots + a_{n-1} e_{n-1}$. Then $L_n T_{n-1}^{-1} U_n = l^M_n a_{n-M} + \cdots + l^1_n a_{n-1}$. Therefore

$$T_n^{-1} e_n = \bar{\mathbf{x}} = \frac{-(a_1 e_1 + \cdots + a_{n-1} e_{n-1}) + e_n}{d_n - (l^M_n a_{n-M} + \cdots + l^1_n a_{n-1})}$$

$$= \begin{bmatrix} -a_1 \\ \vdots \\ -a_{n-1} \\ 1 \end{bmatrix} \frac{1}{d_n - (l^M_n a_{n-M} + \cdots + l^1_n a_{n-1})}. \tag{2.4}$$

In a similar fashion we can calculate $T_n^{*^{-1}} e_n$. We have

$$T_n^* = \begin{bmatrix} T^*_{n-1} & L^*_n \\ U^*_n & d_n \end{bmatrix},$$

where L_n^*, U_n^* denote the adjoint of L_n and U_n respectively. Now, considering the expression

$$T_{n-1}^{*^{-1}} L_n^* = b_1 e_1 + \cdots + b_{n-1} e_{n-1}, \tag{2.5}$$

as in the case of $T_n^{-1} e_n$, we can show that

$$T_n^{*^{-1}} e_n = \frac{-(b_1 e_1 + \cdots + b_{n-1} e_{n-1}) + e_n}{\overline{d}_n - (\overline{u}_n^M b_{n-M} + \cdots + \overline{u}_n^1 b_{n-1})}. \tag{2.6}$$

Now we wish to calculate $T_n^{-1} e_{n-1}$. Again let $T_n \bar{\mathbf{x}} = e_{n-1}$. For $\bar{\mathbf{x}} = \begin{bmatrix} \bar{\mathbf{x}}_{n-1} \\ x_n \end{bmatrix}$, we have

$$\begin{bmatrix} T_{n-1} & U_n \\ L_n & d_n \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{e}_{n-1} \\ 0 \end{bmatrix},$$

which reduces to

$$T_{n-1} \bar{\mathbf{x}}_{n-1} + U_n x_n = \tilde{e}_{n-1}, \tag{2.7}$$

$$L_n \bar{\mathbf{x}}_{n-1} + d_n x_n = 0, \tag{2.8}$$

where $\tilde{e}_{n-1}$ is the $n-1$ dimensional unit vector, more precisely,

$$\tilde{e}_{n-1}(j) = \begin{cases} 1 & \text{if } j = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Once again, solving, we end up with

$$T_n^{-1} e_{n-1} = \begin{bmatrix} T_{n-1}^{-1} \tilde{e}_{n-1} + \frac{(T_{n-1}^{-1} U_n) L_n T_{n-1}^{-1} \tilde{e}_{n-1}}{d_n - L_n T_{n-1}^{-1} U_n} \\ -\frac{L_n T_{n-1}^{-1} \tilde{e}_{n-1}}{d_n - L_n T_{n-1}^{-1} U_n} \end{bmatrix}.$$

Similarly, we obtain

$$T_n^{*^{-1}} e_{n-1} = \begin{bmatrix} T_{n-1}^{*^{-1}} \tilde{e}_{n-1} + \frac{(T_{n-1}^{*^{-1}} L_n^*) U_n^* T_{n-1}^{*^{-1}} \tilde{e}_{n-1}}{\overline{d}_n - U_n^* T_{n-1}^{*^{-1}} L_n^*} \\ -\frac{U_n^* T_{n-1}^{*^{-1}} \tilde{e}_{n-1}}{\overline{d}_n - U_n^* T_{n-1}^{*^{-1}} L_n^*} \end{bmatrix}. \tag{2.9}$$

Similarly, we can calculate $T_n^{*^{-1}} e_{n-2}, \ldots, T_n^{*^{-1}} e_{n-M}$ and $T_n^{-1} e_{n-2}, \ldots, T_n^{-1} e_{n-M}$. Now, let $T_n x^n = y_n$ and $x^n = \alpha_1^{(n)} e_1 + \cdots + \alpha_n^{(n)} e_n$. Then

$$\alpha_n^{(n)} = \langle x^n, e_n \rangle = \left\langle T_n^{-1} y_n, e_n \right\rangle = \left\langle y_n, T_n^{*^{-1}} e_n \right\rangle. \tag{2.10}$$

In general, we find

$$\alpha_{n-j}^{(n)} = \langle x^n, e_{n-j} \rangle, \tag{2.11}$$

for $0 \leq j \leq M-1$. In the next proposition, we show that if $\{T_n^{*^{-1}} e_{n-j}\}$ is bounded in norm for $0 \leq j \leq M-1$, then $\{\alpha_{n-j}^{(n)}\}$ is a member of $\ell^2(\mathbb{N})$.

In order to avoid notational complexity, hereafter we assume that the entries of T are real numbers.

Proposition 2.1 *For $M \in \mathbb{N}$ fixed, let T be a $2M+1$ diagonal operator defined as in (2.1). Suppose that T_n is invertible for all n and that there exist constants $K_j > 0$ such that $\left\| T_n^{*^{-1}} e_{n-j} \right\| \leq K_j$ for all n and $0 \leq j \leq M-1$. Then $\{\alpha_{n-j}^{(n)}\} \in \ell^2(\mathbb{N})$ for $0 \leq j \leq M-1$, where $\alpha_{n-j}^{(n)}$ is defined in (2.11).*

Proof Let $y = \sum_{i=1}^{\infty} \beta_i e_i \in \ell^2(\mathbb{N})$. If $y_n = \sum_{i=1}^{n} \beta_i e_i$, then from (2.10) it follows that

$$\alpha_n^{(n)} = \sum_{i=1}^{n} \beta_i \left\langle e_i, T_n^{*^{-1}} e_n \right\rangle. \tag{2.12}$$

Using relation (2.6) we can write

$$\alpha_n^{(n)} = \sum_{i=1}^{n} \beta_i \left\langle e_i, \frac{-(b_1 e_1 + \cdots + b_{n-1} e_{n-1}) + e_n}{d_n - (u_n^M b_{n-M} + \cdots + u_n^1 b_{n-1})} \right\rangle.$$

In other words, $\alpha_n^{(n)} = \sum_{i=1}^{n-1} \frac{\beta_i b_i}{B_n} - \frac{\beta_n}{B_n}$, where $B_n = u_n^M b_{n-M} + \cdots + u_n^1 b_{n-1} - d_n$. Then

$$|\alpha_n^{(n)}| \leq \sum_{i=1}^{n-1} \frac{|\beta_i b_i|}{|B_n|} + \frac{|\beta_n|}{|B_n|}.$$

By the given hypothesis, $\left\| T_n^{*^{-1}} e_n \right\| \leq K_0$ for all n. Then, from (2.6), it follows that

$$1 + b_1^2 + \cdots + b_{n-1}^2 \leq K_0^2 B_n^2. \tag{2.13}$$

Using Lemma 4.2 in [10], and taking $r = 8$, we show that

$$\frac{|b_i|}{|B_n|} \leq \frac{C}{(n-i-1)^4},$$

where C is a constant and $1 \leq i \leq n-1$. Therefore

$$\begin{aligned}
|\alpha_n^{(n)}| &\leq C \sum_{i=1}^{n-1} \frac{|\beta_i|}{(n-i-1)^4} + \frac{|\beta_n|}{|B_n|} \\
&\leq C \sum_{i=1}^{\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} + C \sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} + \frac{|\beta_n|}{|B_n|}.
\end{aligned}$$

For $i \leq \frac{n-1}{2}$, $n-i-1 \geq \frac{n-1}{2}$ and so $\frac{1}{n-i-1} \leq \frac{2}{n-1}$.

$$|\alpha_n^{(n)}| \leq 2^4 C \sum_{i=1}^{\frac{n-1}{2}} \frac{|\beta_i|}{(n-1)^4} + C \sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} + \frac{|\beta_n|}{|B_n|}.$$

By using the Cauchy Schwarz inequality, we deduce

$$\begin{aligned}
|\alpha_n^{(n)}| &\leq \frac{D}{(n-1)^4} \left[\|y\|_{\ell^2} \left(\sum_{i=1}^{\frac{n-1}{2}} 1 \right)^{1/2} \right] + C \sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} + \frac{|\beta_n|}{|B_n|} \\
&\leq \frac{D}{(n-1)^4} \left[\frac{(n-1)}{2} \right]^{1/2} \|y\|_{\ell^2} + C \sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} + \frac{|\beta_n|}{|B_n|} \\
&= \mathcal{O}\left(\frac{1}{n^{7/2}} \right) + C \sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} + \frac{|\beta_n|}{|B_n|}.
\end{aligned}$$

Consider $P \leq n < N$. We have

$$\begin{aligned}
&\sum_{P \leq n < N} \left(\sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} + \frac{|\beta_n|}{|B_n|} \right)^2 \\
&= \sum_{P \leq n < N} \left[\left(\sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} \right)^2 + \frac{|\beta_n|^2}{|B_n|^2} + 2 \sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} \frac{|\beta_n|}{|B_n|} \right].
\end{aligned}$$

Thus

$$\sum_{P\le n<N}|\alpha_n^{(n)}|^2 = C\sum_{P\le n<N}\left[\sum_{i_1,i_2>\frac{n-1}{2}}\frac{|\beta_{i_1}||\beta_{i_2}|}{(n-i_1-1)^4(n-i_2-1)^4}\right.$$
$$\left.+\frac{|\beta_n|^2}{|B_n|^2}+\sum_{i>\frac{n-1}{2}}\frac{|\beta_i|}{(n-i-1)^4}\frac{|\beta_n|}{|B_n|}\right]+\sum_{P\le n<N}\frac{1}{n^7}$$
$$+2\sum_{P\le n<N}\left(\frac{1}{n^{7/2}}\sum_{i>\frac{n-1}{2}}\left(\frac{|\beta_i|}{(n-i-1)^4}+\frac{|\beta_n|}{|B_n|}\right)\right)$$
$$= C\sum_{\substack{i_1,i_2>\frac{n-1}{2}\\ P\le n<N}}\frac{|\beta_{i_1}||\beta_{i_2}|}{(n-i_1-1)^4(n-i_2-1)^4}$$
$$+\sum_{P\le n<N}\frac{|\beta_n|^2}{|B_n|^2}+\sum_{\substack{i>\frac{n-1}{2}\\ P\le n<N}}\frac{|\beta_i|}{(n-i-1)^4}\frac{|\beta_n|}{|B_n|}+\sum_{P<n<N}\frac{1}{n^7}$$
$$+2\sum_{P\le n<N}\left(\frac{1}{n^{7/2}}\sum_{i>\frac{n-1}{2}}\left(\frac{|\beta_i|}{(n-i-1)^4}+\frac{|\beta_n|}{|B_n|}\right)\right). \tag{2.14}$$

Further from (2.13), $\frac{1}{|B_n|^2}\le K_0^2$. Therefore

$$\sum_{P\le n<N}\frac{|\beta_n|^2}{|B_n|^2}\le K_0^2\sum_{P\le n<N}|\beta_n|^2\to 0,$$

as $P, N\to\infty$ and $\sum_{P\le n<N}\frac{1}{n^7}\to 0$ as $P, N\to\infty$. Now, consider the sum

$$\sum_{\substack{i>\frac{n-1}{2}\\ P\le n<N}}\frac{|\beta_i|}{(n-i-1)^4}\frac{|\beta_n|}{|B_n|}\le K_0\sum_{\substack{i>\frac{n-1}{2}\\ P\le n<N}}\frac{|\beta_i||\beta_n|}{(n-i-1)^4}.$$

By the Cauchy–Schwarz inequality, we get

$$\sum_{P\le n<N}\sum_{i>\frac{n-1}{2}}\frac{|\beta_i||\beta_n|}{(n-i-1)^4}$$

$$\leq \left(\sum_{P<n<N} \left(\sum_{i>\frac{n-1}{2}} \frac{|\beta_i|}{(n-i-1)^4} \right)^2 \right)^{1/2} \left(\sum_{P\leq n<N} |\beta_n|^2 \right)^{1/2}$$

$$\leq \left(\sum_{\substack{i_1,i_2>\frac{n-1}{2} \\ P\leq n<N}} \frac{|\beta_{i_1}||\beta_{i_2}|}{(n-i_1-1)^4(n-i_2-1)^4} \right)^{\frac{1}{2}} \|y\|_{\ell^2}$$

$$\leq \left(\sum_{\substack{i_1,i_2>\frac{n-1}{2} \\ P\leq n<N}} \frac{|\beta_{i_1}|^2+|\beta_{i_2}|^2}{(n-i_1-1)^4(n-i_2-1)^4} \right)^{\frac{1}{2}} \|y\|_{\ell^2}\,.$$

Now

$$\sum_{\substack{i_1,i_2>\frac{n-1}{2} \\ P\leq n<N}} \frac{|\beta_{i_1}|^2}{(n-i_1-1)^4(n-i_2-1)^4}$$
$$\leq \sum_{P\leq i_1<N} |\beta_{i_1}|^2 \sum_{i_2>i_1} \frac{1}{(i_2-i_1-1)^4} \sum_{n>i_2} \frac{1}{(n-i_2-1)^4}.$$

Since $\{\beta_i\} \in \ell^2(\mathbb{N})$, from (2.14) and the inequality $2ab \leq a^2+b^2$ it follows that $\{\alpha_n^{(n)}\} \in \ell^2(\mathbb{N})$. We now show that $\{\alpha_{n-1}^{(n)}\} \in \ell^2(\mathbb{N})$. From (2.11) we have

$$\alpha_{n-1}^{(n)} = \left\langle x^n, e_{n-1} \right\rangle = \sum_{i=1}^{n} \beta_i \left\langle e_i, T_n^{*^{-1}} e_{n-1} \right\rangle. \tag{2.15}$$

Let $T_{n-1}^{*^{-1}} \tilde{e}_{n-1} = c_1 e_1 + \cdots + c_{n-1} e_{n-1}$. Then

$$U_n^* T_{n-1}^{*^{-1}} \tilde{e}_{n-1} = (0, \cdots, 0, u_{n-M}^M, \cdots, u_{n-1}^1)(c_1, \cdots, c_{n-1})^T$$
$$= c_{n-M} u_{n-M}^M + \cdots + u_{n-1}^1 c_{n-1}.$$

Further let $D_n = c_{n-M} u_{n-M}^M + \cdots + u_{n-1}^1 c_{n-1}$ and $A_n = d_n - U_n^* T_{n-1}^{*^{-1}} L_n^*$. Then, using (2.5) and (2.9), we can write

$$T_n^{*^{-1}} e_{n-1} = \begin{bmatrix} c_1 + \frac{D_n b_1}{A_n} \\ \vdots \\ c_{n-1} + \frac{D_n b_{n-1}}{A_n} \\ -\frac{D_n}{A_n} \end{bmatrix}.$$

Thus by (2.15)

$$\alpha_{n-1}^{(n)} = \sum_{i=1}^{n} \beta_i \left\langle e_i, \left(c_1 + \frac{D_n b_1}{A_n}\right) e_1 + \cdots, + \left(c_{n-1} + \frac{D_n b_{n-1}}{A_n}\right) e_{n-1} - \frac{D_n}{A_n} e_n \right\rangle$$
$$= \sum_{i=1}^{n-1} \beta_i \left(c_i + \frac{D_n b_i}{A_n}\right) - \beta_n \frac{D_n}{A_n}.$$

So, $|\alpha_{n-1}^{(n)}| \leq \sum_{i=1}^{n-1} \left|\beta_i \left(c_i + \frac{D_n b_i}{A_n}\right)\right| + |\beta_n| \left|\frac{D_n}{A_n}\right|$. Since $\left\|T_n^{*^{-1}} e_{n-1}\right\| \leq K_1$ for all n, we obtain

$$\left|c_1 + \frac{D_n b_1}{A_n}\right|^2 + \cdots + \left|c_{n-1} + \frac{D_n b_{n-1}}{A_n}\right|^2 + \frac{|D_n|^2}{|A_n|^2} \leq K_1^2. \tag{2.16}$$

Also $c_1^2 + \cdots + c_{n-1}^2 \leq K_0^2$. Let $h_{i,n} = c_i A_n + D_n b_i$. Then

$$|\alpha_{n-1}^{(n)}| \leq \sum_{i=1}^{n-1} \left|\frac{\beta_i h_{i,n}}{A_n}\right| + |\beta_n| \left|\frac{D_n}{A_n}\right|.$$

Furthermore, since the sequence $\{A_n\}$ is bounded below, we can write $1 + h_{1,n}^2 + \cdots + h_{n-1,n}^2 \leq K^2 \|A_n\|^2$, for some constant K. Now, we employ Lemma 4.2 in [10] and, proceeding as before, it can be shown that $\{\alpha_{n-1}^{(n)}\} \in \ell^2(\mathbb{N})$. In a similar fashion, we can show that $\{\alpha_{n-j}^{(n)}\} \in \ell^2(\mathbb{N})$ for $0 \leq j \leq M - 1$. □

Now we are in a position to prove our main result.

3 The main result

Theorem 3.1 *Let T be a $2M + 1$ diagonal operator defined by (2.1). Suppose that T_n is invertible for all n and that there exist constants K_l such that $\left\|T_n^{-1} e_{n-l}\right\| \leq K_l$ for all $0 \leq l \leq M - 1$ and n. If y is in the range of T, then the solution of the operator equation $Tx = y$ can be obtained as the limit of the solutions x^n of the operator equation $T_n x^n = y_{n|_{H_n}}$ in the norm topology. In particular T is one-one. In addition, if there exist constants $K_l^{'}$ such that $\left\|T_n^{*^{-1}} e_{n-l}\right\| \leq K_l^{'}$ for all n and $0 \leq l \leq M - 1$, then T is onto and hence invertible.*

Proof Let $y \in R(T)$, where $R(T)$ denotes the range of operator T. Let $Tx = y$. We write $x = \sum_{i=1}^{\infty} \alpha_i e_i$, $x_n = \sum_{i=1}^{n} \alpha_i e_i$ and $y_n = P_n y$. Then we have $\langle T_n(x_n), e_n\rangle = \alpha_{n-M} l_{n+M}^{M} + \cdots + \alpha_{n-1} l_{n+1}^{1} + \alpha_n d_n$. On the other hand $\langle T(x), e_n\rangle =$

$\sum_{j=1}^{M} \alpha_{n-j} u_{n-j}^{j} + \alpha_n d_n + \sum_{j=1}^{M} \alpha_{n+j} l_{n+j}^{j}$. Then we write for large n

$$T_n(x_n) + \sum_{k=1}^{M} \alpha_{n+k} u_n^k e_n = y_n.$$

As $n \to \infty, x_n \to x$. Further $\{u_n^k\}$ is a bounded sequence, $\{\alpha_{n+k}\} \in \ell^2(\mathbb{N})$ and $\left\|T_n^{-1} e_n\right\| \leq K_0$ which show that $T_n^{-1} y_n \to x$. In particular if $Tx = 0$, then $y = 0$ which in turn implies that $y_n = P_n y = 0$. Hence $T_n^{-1} y_n = 0$, which shows that $x = 0$. Thus T is one-one.

We now prove that T is onto. Let $y \in \mathcal{H}$. Then $y = \sum_{i=1}^{\infty} \beta_i e_i$ with $\sum_{i=1}^{\infty} |\beta_i|^2 < \infty$. We write $y_n = \sum_{i=1}^{n} \beta_i e_i$. Since each T_n is onto there exists $x^n \in \mathcal{H}_n$ such that $T_n x^n = y_n$, we can write $x^n = \alpha_1^n e_1 + \cdots + \alpha_n^n e_n$. Further $T(x^n) = \alpha_1^n T e_1 + \cdots + \alpha_n^n T e_n$, from which it follows that

$$T(x^n) = T_n(x^n) + \alpha_{n-M+1}^n l_{n+1}^M e_{n+1} + \cdots + \alpha_n^n \sum_{j=1}^{M} l_{n+j}^j e_{n+j}. \tag{3.1}$$

So $T(x^n)$ and $T_n(x^n)$ differ only by the terms $\alpha_{n-M+1}^n l_{n+1}^M e_{n+1} + \cdots + \alpha_n^n \sum_{j=1}^{M} l_{n+j}^j e_{n+j}$. Hence

$$\left\|T_n(x^n) - T(x^n)\right\|^2 \leq K^2 \left(|\alpha_{n-M+1}^n|^2 + \cdots + |\alpha_n^n|^2\right) \to 0$$

by Proposition 2.1. Now, if we show that $\{x^n\}$ is a Cauchy sequence in $\mathcal{H}$, then there exists $x \in \mathcal{H}$ such that $x^n \to x$ in $\mathcal{H}$. Since T is continuous $T(x^n) \to T(x)$ and in the limit $T(x^n)$ and $T_n(x^n)$ coincide by (3.1). Also

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} T_n(x^n) = \lim_{n\to\infty} T(x^n) = T(x),$$

showing that T is onto. □

Lemma 3.2 *$\{x^n\}$ is a Cauchy sequence.*

Proof Consider $x^{n+1} - x^n = T_{n+1}^{-1} y_{n+1} - T_{n+1}^{-1} T_{n+1} x^n$. We know that $x^n = \alpha_1^n e_1 + \cdots + \alpha_n^n e_n$. Hence

$$T_{n+1}(x^n) = T_n(x^n) + \alpha_{n-M+1}^n l_{n+1}^M e_{n+1} + \cdots + \alpha_n^n l_{n+1}^1 e_{n+1}.$$

So

$$T_{n+1}(x^n) = y_n + \alpha_{n-M+1}^n l_{n+1}^M e_{n+1} + \cdots + \alpha_n^n l_{n+1}^1 e_{n+1}.$$

Therefore

$$x^{n+1} - x^n = T_{n+1}^{-1} y_{n+1} - T_{n+1}^{-1} y_n - \alpha_{n-M+1}^n l_{n+1}^M T_{n+1}^{-1} e_{n+1} - \cdots - \alpha_n^n l_{n+1}^1 T_{n+1}^{-1} e_{n+1}.$$

Let $y = \sum_{i=1}^{\infty} \beta_i e_i$, then $y_{n+1} - y_n = \beta_{n+1} e_{n+1}$. Thus $T_{n+1}^{-1}(y_{n+1} - y_n) = \beta_{n+1} T_{n+1}^{-1} e_{n+1}$. Hence

$$\begin{aligned} x^{n+1} - x^n &= \beta_{n+1} T_{n+1}^{-1} e_{n+1} - \alpha_{n-M+1}^n l_{n+1}^M T_{n+1}^{-1} e_{n+1} - \cdots - \alpha_n^n l_{n+1}^1 T_{n+1}^{-1} e_{n+1} \\ &= \left(\beta_{n+1} - \alpha_{n-M+1}^n l_{n+1}^M - \cdots - \alpha_n^n l_{n+1}^1\right) T_{n+1}^{-1} e_{n+1}. \end{aligned}$$

For $P > N$

$$x^P - x^N = \sum_{N \le n < P} \left(\beta_{n+1} - \alpha_{n-M+1}^n l_{n+1}^M - \cdots - \alpha_n^n l_{n+1}^1\right) T_{n+1}^{-1} e_{n+1}.$$

Let $\gamma_n = \left(\beta_{n+1} - \alpha_{n-M+1}^n l_{n+1}^M - \cdots - \alpha_n^n l_{n+1}^1\right)$. Since $\{\beta_n\} \in \ell^2(\mathbb{N})$ and by Proposition 2.1 $\{\alpha_{n-l}^n\} \in \ell^2(\mathbb{N})$ for $0 \le l \le M-1$, $\gamma_n \in \ell^2(\mathbb{N})$. Then $x^P - x^N = \sum_{N \le n < P} \gamma_n T_{n+1}^{-1} e_{n+1}$. Using (2.4), we deduce

$$T_{n+1}^{-1} e_{n+1} = \frac{1}{E_{n+1}} \left(a_1 e_1 + \cdots + a_n e_n - e_{n+1}\right),$$

where $E_{n+1} = \frac{1}{l_{n+1}^M a_{n+1-M} + \cdots + l_{n+1}^1 a_n - d_{n+1}}$. Thus

$$x^P - x^N = \sum_{N \le n < P} \gamma_n \left[\sum_{i=1}^{n+1} \frac{a_i e_i}{E_{n+1}}\right],$$

where $a_{n+1} = -1$. We set $a_i = 0$ for $i > n+1$. Hence

$$x^P - x^N = \sum_{i=1}^{\infty} \sum_{N \le n < P} \gamma_n \frac{a_i e_i}{E_{n+1}}.$$

Therefore

$$\begin{aligned} \left\| x^P - x^N \right\|^2 &= \sum_{i=1}^{\infty} \left| \sum_{N \le n < P} \frac{\gamma_n}{E_{n+1}} \right|^2 a_i^2 \le \sum_{i=1}^{\infty} \sum_{\substack{n_1, n_2 \\ N \le n_1, n_2 < P}} \frac{|\gamma_{n_1}||\gamma_{n_2}|}{|E_{n_1+1}||E_{n_2+1}|} a_i^2 \\ &= \sum_{\substack{n_1, n_2 \\ N \le n_1, n_2 < P}} \frac{|\gamma_{n_1}||\gamma_{n_2}|}{|E_{n_1+1}||E_{n_2+1}|} \sum_{i=1}^{n_1+1} a_i^2. \end{aligned}$$

By the given assumptions, $\left\| T_n^{-1} e_n \right\| \le K_0$ for all n implies

$$\sum_{i=1}^{n_1+1} a_i^2 \le K_0^2 \left(E_{n_1+1}\right)^2.$$

Then

$$\left\| x^P - x^N \right\| \leq 2K_0^2 \sum_{n_1 \leq n_2} \frac{|\gamma_{n_1}||\gamma_{n_2}||E_{n_1+1}|}{|E_{n_2+1}|}$$
$$= 2K_0^2 \sum_{n_1} |\gamma_{n_1}|^2 + 2K_0^2 \sum_{n_1 < n_2} \frac{|\gamma_{n_1}||\gamma_{n_2}||E_{n_1+1}|}{|E_{n_2+1}|}.$$

Consider the sum, $\sum_{n_1<n_2} \frac{|\gamma_{n_1}||\gamma_{n_2}||E_{n_1+1}|}{|E_{n_2+1}|} \leq \sum_{n_1<n_2} \frac{(|\gamma_{n_1}|^2+|\gamma_{n_2}|^2)|E_{n_1+1}|}{|E_{n_2+1}|}$. Now we can use the same technique as in Proposition 2.1, to prove that

$$\left\| x^P - x^N \right\| \to 0$$

as $P, N \to \infty$, establishing that $\{x^n\}$ is a Cauchy sequence. □

4 A verifiable condition for the invertibility of a tridiagonal operator on $\ell^2(\mathbb{N})$

The tridiagonal operator T can be written as

$$Te_1 = d_1 e_1 + l_2 e_2$$
$$Te_n = u_{n-1} e_{n-1} + d_n e_n + l_{n+1} e_{n+1} \quad n \geq 2.$$

In particular,

$$T_n = \begin{bmatrix} d_1 & u_1 & 0 & 0 & \cdots & 0 & 0 \\ l_2 & d_2 & u_2 & 0 & \cdots & 0 & 0 \\ 0 & l_3 & d_3 & u_3 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & d_{n-1} & u_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & l_n & d_n \end{bmatrix}$$

Here we can write $U_n = u_{n-1} e_{n-1}$, $L_n = l_n e_{n-1}^T$. Recall from Section 1

$$T_n^{-1} e_n = \frac{[T_{n-1}^{-1} U_n \quad 0]^T - e_n}{L_n T_{n-1}^{-1} U_n - d_n} = \frac{[u_{n-1} T_{n-1}^{-1} e_{n-1} \quad 0]^T - e_n}{l_n u_{n-1} e_{n-1}^T T_{n-1}^{-1} e_{n-1} - d_n}. \tag{4.1}$$

Let $R_{n-1} = e_{n-1}^T T_{n-1}^{-1} e_{n-1}$. Hence we obtain the recurrence for R_n, given by

$$R_n = \frac{1}{l_n u_{n-1} R_{n-1} - d_n} \; \forall\, n \geq 2. \tag{4.2}$$

Proposition 4.1 *Let $\{d_n\}$, $\{u_n\}$ and $\{l_n\}$ be bounded sequences, bounded from above by d, u and c respectively. Let m_0 be a number satisfying $0 < m_0 < \frac{1}{c^2}$, $m_0 < \frac{1}{uc}$ such that $|R_n R_{n-1}| \leq m_0$ for all n. Then $\left\{\left\| T_n^{-1} e_n \right\|\right\}$ is bounded.*

Proof We can write $T_n^{-1} e_n = R_n \left(\begin{bmatrix} u_{n-1} T_{n-1}^{-1} e_{n-1} & 0 \end{bmatrix}^T - e_n \right)$. So

$$\left\| T_n^{-1} e_n \right\| \leq |R_n| + |R_n||u_{n-1}| \left\| T_{n-1}^{-1} e_{n-1} \right\|.$$

Again we can use the recursive relation (4.2) to infer

$$\left\| T_n^{-1} e_n \right\| \leq |R_n| + |R_n||u_{n-1}| \left\| R_{n-1} \left(\begin{bmatrix} u_{n-2} T_{n-2}^{-1} e_{n-2}. & 0 \end{bmatrix}^T - e_{n-1} \right) \right\|.$$

In other words

$$\left\| T_n^{-1} e_n \right\| \leq |R_n| + |R_n||R_{n-1}||u_{n-1}| + |R_n||R_{n-1}||u_{n-1}||u_{n-2}| \left\| T_{n-2}^{-1} e_{n-2} \right\|.$$

Continuing this way, we obtain

$$\left\| T_n^{-1} e_n \right\| \leq |R_n| \left(1 + |R_{n-1}||u_{n-1}| + \cdots + \frac{|R_{n-1}| \cdots |R_2||u_{n-1}||u_{n-2}| \cdots |u_2|}{d_1} \right).$$

Then

$$\begin{aligned} \left\| T_n^{-1} e_n \right\| &\leq |R_n| \left(1 + m_0 u + m_0 u^2 + m_0^2 u^3 + m_0^2 u^4 + \cdots + m_0^{\frac{n-2}{2}} u^{n-2} \right) \\ &\leq |R_n| \left(1 + m_0 u^2 + m_0^2 u^4 + \cdots \right) + m_0 u + m_0^2 u^3 + \cdots \\ &= |R_n| \left(\frac{1}{1 - m_0 u^2} \right) + m_0 u (1 + m_0 u^2 + m_0^2 u^4 + \cdots) \\ &= (m_0 u + |R_n|) \frac{1}{1 - m_0 u^2}. \end{aligned}$$

Since $|R_n R_{n-1}| \leq m_0$, $|R_n| \leq \frac{|d_n|}{1 - m_0 |l_n||u_{n-1}|}$. But $|l_n||u_{n-1}| \leq uc$ and $m_0 < \frac{1}{uc}$. Hence $\left\| T_n^{-1} e_n \right\| \leq \left(m_0 u + \frac{d}{1 - m_0 uc} \right) \frac{1}{1 - m_0 u^2}$, showing that $\{T_n^{-1} e_n\}$ is bounded in norm. □

In the next result, we address the question "how the entries d_n, u_n, l_n can be choosen so that $|R_n R_{n+1}| \leq m_0$ for all n".

Theorem 4.2 *If $|d_n d_{n+1}| \geq \frac{(1 + m_0 |u_n l_{n+1}|)(1 - m_0 |u_{n-1} l_n|)}{m_0 (1 - 2 m_0 |u_{n-1} l_n|)}$, where $0 < m_0 < \frac{1}{c^2}$, $m_0 < \frac{1}{2uc}$, u, c are upper bounds of $\{u_n\}$ and $\{l_n\}$, respectively, then $\{\left\| T_n^{-1} e_n \right\|\}$ is bounded.*

Proof By Proposition 4.1, we need to show that $R_n R_{n+1} \leq m_0 \ \forall \ n$. Recall $R_n = \frac{1}{l_n u_{n-1} R_{n-1} - d_n}$. We observe the following $R_1 = e_1^T T_1^{-1} e_1 = \frac{1}{d_1}$ and $R_2 = \frac{1}{d_2 - l_2 u_1 R_1} = \frac{d_1}{d_1 d_2 - l_2 u_1} = \frac{\det T_1}{\det T_2}$. By induction we want to show that $R_n = \frac{\det T_{n-1}}{\det T_n}$. Assume that it is true for $n = k$. In other words,

$$R_k = \frac{\det T_{k-1}}{\det T_k}.$$

Consider

$$\begin{aligned} R_{k+1} &= \frac{1}{d_{k+1} - l_{k+1} u_k R_k} \\ &= \frac{1}{d_{k+1} - l_{k+1} u_k \frac{\det T_{k-1}}{\det T_k}} = \frac{\det T_k}{d_{k+1} \det T_k - l_{k+1} u_k \det T_{k-1}}. \end{aligned}$$

By the recurrence relation $\det T_{k+1} = d_{k+1} \det T_k - l_{k+1} u_k \det T_{k-1}$ (see (1) in [10]), we conclude that

$$R_{k+1} = \frac{\det T_k}{\det T_{k+1}}.$$

(We can also obtain this expression for R_n using Cramer's rule). We now prove by induction $|R_n R_{n+1}| < m_0 \ \forall \ n$. For $n = 1$

$$|R_1 R_2| = \frac{1}{d_1 d_2 - u_2 l_1}.$$

Assume that $|R_n R_{n-1}| \leq m_0$. In order to show that $|R_n R_{n+1}| < m_0$, we consider

$$\begin{aligned} |\det T_{n+1}| &= |d_{n+1} \det T_n - l_{n+1} u_n \det T_{n-1}| \\ &= |d_{n+1} \{d_n \det T_{n-1} - l_n u_{n-1} \det T_{n-2}\} - l_{n+1} u_n \det T_{n-1}| \\ &= |d_n d_{n+1} \det T_{n-1} - d_{n+1} l_n u_{n-1} \det T_{n-2} - l_{n+1} u_n \det T_{n-1}| \\ &\geq |d_n d_{n+1} \det T_{n-1}| - |d_{n+1} l_n u_{n-1} \det T_{n-2}| - |l_{n+1} u_n \det T_{n-1}| \\ &= |d_n d_{n+1}||\det T_{n-1}| - |l_{n+1} u_n||\det T_{n-1}| - |d_{n+1} l_n u_{n-1} \det T_{n-2}|. \end{aligned}$$

Dividing by $|\det T_{n-1}|$ on both sides, we get

$$\frac{|\det T_{n+1}|}{|\det T_{n-1}|} \geq |d_n d_{n+1}| - |l_{n+1} u_n| - |d_{n+1} l_n u_{n-1}| \frac{|\det T_{n-2}|}{|\det T_{n-1}|}.$$

As $R_n = \frac{1}{l_n u_{n-1} R_{n-1} - d_n}$, $|R_n R_{n-1}| \leq m_0$, we have $\left| \frac{R_{n-1}}{l_n u_{n-1} R_{n-1} - d_n} \right| \leq m_0$. In other words, $|R_{n-1}| \leq m_0 |d_n| + m_0 |l_n u_{n-1}||R_{n-1}|$ from which it follows that

$$|R_{n-1}| \leq \frac{m_0 |d_n|}{1 - m_0 |l_n u_{n-1}|}.$$

As $R_n = \frac{\det T_{n-1}}{\det T_n}$, $R_{n+1} = \frac{\det T_n}{\det T_{n+1}}$, we have $R_n R_{n+1} = \frac{\det T_{n-1}}{\det T_{n+1}}$. Thus

$$\begin{aligned}\frac{|\det T_{n+1}|}{|\det T_{n-1}|} &\geq |d_n d_{n+1}| - |l_{n+1}u_n| - |d_{n+1}l_n u_{n-1}|\left(\frac{m_0|d_n|}{1-m_0|l_n u_{n-1}|}\right)\\ &= |d_n d_{n+1}|\left(1 - \frac{m_0|l_n u_{n-1}|}{1-m_0|l_n u_{n-1}|}\right) - |l_{n+1}u_n|\\ &\geq \frac{1}{m_0}\end{aligned}$$

which shows that $|R_n R_{n+1}| \leq m_0$, proving our assertion. □

Corollary 4.3 *Let T be the tridiagonal operator with diagonal entries $\{d_n\}$ and off diagonal entries 1 . If the sequence $\{d_n\}$ satisfies $|d_n d_{n+1}| \geq \frac{1-m_0^2}{m_0(1-2m_0)}$ for some m_0 satisfying $0 < m_0 < \frac{1}{2}$. Then $\{\|T_n^{-1}e_n\|\}$ is bounded.*

Proof Take $u_n = l_n = 1$ in Theorem 4.2. □

Example 4.1 Let $u_n = l_n = 1$ for all $n \in \mathbb{N}$ and

$$d_n = \begin{cases} 1, & \text{if } n \text{ is odd,}\\ 10, & \text{if } n \text{ is even.}\end{cases}$$

Let $m_0 = \frac{1}{4}$. Then $\frac{1-m_0^2}{m_0(1-2m_0)} = 7.5$, and we have $|d_n d_{n+1}| = 10 > 7.5$. Then, from Corollary 4.3, it follows that $\{T_n^{-1}e_n\}$ is bounded in norm. Since $T_n^* = T_n$, the sequence $\{\|T_n^{*^{-1}}e_n\|\}$ is also bounded. This leads to the invertibility of T (Theorem 3.1).

Example 4.2 Let $u_n = 2,\ l_n = 3$, for all $n \in \mathbb{N}$ and

$$d_n = \begin{cases} 4, & \text{if } n \text{ is odd,}\\ 19, & \text{if } n \text{ is even.}\end{cases}$$

Let $m_0 = \frac{1}{18}$. Then $\frac{(1-6m_0)(1+6m_0)}{m_0(1-12m_0)} = 72$, and we have $|d_n d_{n+1}| = 76 > 72$, $m_0 < \frac{1}{c^2}$, $m_0 < \frac{1}{2uc}$. Then, from Proposition 4.1, it follows that $|R_n R_{n+1}| < \frac{1}{18}$. From Theorem 4.2 it follows that $\{\|T_n^{-1}e_n\|\}$ is bounded. Since $T_n^* = T_n$, we have $\|T_n^{*^{-1}}e_n\|$ is also bounded. This leads to the invertibility of T.

Example 4.3 Let $u_n = l_n = 1$, for all $n \in \mathbb{N}$ and

$$d_n = \begin{cases} n, & \text{if } n \text{ is odd,}\\ \frac{10}{n-1}, & \text{if } n \text{ is even.}\end{cases}$$

Let $m_0 = \frac{1}{4}$. Then $\frac{1-m_0^2}{m_0(1-2m_0)} = 7.5$ and clearly $|d_n d_{n+1}| = 10 > 7.5$. Then, from Corollary 4.3, it follows that $\{T_n^{-1}e_n\}$ is bounded in norm. This leads to the invertibility of T.

5 Extension to the invertibility of a $(2M + 1)$ diagonal operator on $\ell^2(\mathbb{Z})$

Let $V = \ell^2(\mathbb{Z})$ denote a separable Hilbert space. Let $\{\cdots, e_{-n}, , \cdots, e_{-1}, e_0, e_1, \cdots, e_n, \cdots\}$ be the standard orhonormal basis for V. Let

$$V_n = \text{span}\{e_{-n}, \cdots, e_0, \cdots, e_n\}$$

and $P_n : V \to V_n \subseteq V$ be the orthogonal projection on V_n. Let T be a $2M + 1$ diagonal operator defined on $V \simeq \ell^2(\mathbb{Z})$ by

$$Te_n = \sum_{j=1}^{M} u_{n-j}^{j} e_{n-j} + d_n e_n + \sum_{j=1}^{M} l_{n+j}^{j} e_{n+j}, \tag{5.1}$$

where $\{d_n\}$, $\{l_n^j\}$, and $\{u_n^j\}$ are bounded sequences of complex numbers and $T_n = P_n T P_n$. Then

$$\begin{aligned}
T_n e_{-n} &= d_{-n} e_{-n} + \sum_{j=1}^{M} u_{-n+j}^{j} e_{-n+j},\\
T_n e_i &= \sum_{j=1}^{M} u_{i-j}^{j} e_{i-j} + d_i e_i + \sum_{j=1}^{M} l_{i+j}^{j} e_{i+j}, \quad -n+1 \le i \le n-1,\\
T_n e_n &= \sum_{j=1}^{M} l_{n-j}^{j} e_{n-j} + d_n e_n.
\end{aligned}$$

Let $S_n : V \to \text{span}\{e_{-M+1}, \cdots, e_n\}$ and $Q_n : V \to \text{span}\{e_{-n}, \cdots, e_{-M+1}\}$. In other words,

$$T_n = \begin{bmatrix}
d_{-n} & u_{-n}^1 & \cdots & u_{-n}^M & 0 & 0 & \cdots & 0 & 0\\
l_{-n+1}^1 & d_{-n+1} & u_{-n+1}^1 & \cdots & u_{-n+1}^M & 0 & \cdots & 0 & 0\\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\
0 & \cdots & 0 & l_{n-1}^M & \cdots & l_{n-1}^2 & l_{n-1}^1 & d_{n-1} & 0\\
0 & \cdots & 0 & 0 & l_n^M & \cdots & l_n^2 & l_n^1 & d_n
\end{bmatrix},$$

$$S_n = \begin{bmatrix} d_0 & u_0^1 & \cdots & u_0^M & 0 & 0 & \cdots & 0 & 0 \\ l_1^1 & d_1 & u_1^1 & \cdots & u_1^M & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots & \\ 0 & \cdots & 0 & l_{n-1}^M & \cdots & l_{n-1}^2 & l_{n-1}^1 & d_{n-1} & 0 \\ 0 & \cdots & 0 & 0 & l_n^M & \cdots & l_n^2 & l_n^1 & d_n \end{bmatrix},$$

$$Q_n = \begin{bmatrix} d_{-n} & u_{-n}^1 & \cdots & u_{-n}^M & 0 & 0 & \cdots & 0 & 0 \\ l_{-n+1}^1 & d_{-n+1} & u_{-n+1}^1 & \cdots & u_{-n+1}^M & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & l_{-1}^M & \cdots & l_{-1}^2 & l_{-1}^1 & d_{-1} & 0 \\ 0 & \cdots & 0 & 0 & l_0^M & \cdots & l_0^2 & l_0^1 & d_0 \end{bmatrix}.$$

Let $T_n x^{(n)} = y_n, x^{(n)} = \alpha_{-n}^{(n)} e_{-n} + \cdots + \alpha_0^{(n)} e_0 + \cdots + \alpha_n^{(n)} e_n$. Then for l, $\alpha_{n-l}^{(n)} = \langle x^{(n)}, e_{n-l} \rangle$ and $\alpha_{-n+l}^{(n)} = \langle x^{(n)}, e_{-n+l} \rangle$. With these notations, we are in a position to state our main results.

Proposition 5.1 *Let T be a $2M+1$ diagonal operator defined by (5.1). Suppose that Q_n and S_n are invertible for all n and that there exist constants L_l^1 and L_l^2 such that $\left\| S_n^{*^{-1}} e_{n-l} \right\| \leq L_l^1$, $\left\| Q_n^{*^{-1}} e_{-n+l} \right\| \leq L_l^2$ for $0 \leq l \leq M-1$, then $\{\alpha_{n-l}^n\}$ and $\{\alpha_{-n+l}^n\} \in \ell^2(\mathbb{N})$.*

Theorem 5.2 *Let T be a $2M+1$ diagonal operator defined by (5.1). Suppose that T_n, S_n and Q_n are invertible for all n and that there exist constants K_l^1 and K_l^2 such that $\|T_n^{-1} e_{n-l}\| \leq K_l^1$, $\|T_n^{-1} e_{-n+l}\| \leq K_l^2$, for all $0 \leq l \leq M-1$ and n. If y belongs to the range of T, then the solution of the operator equation $Tx = y$ can be obtained as the limit of the solutions $x^{(n)}$ of the operator equation $T_n x^{(n)} = y_{n_{|_{H_n}}}$ in the norm topology. In particular T is one-one. In addition, if $\{u_n^j\}$, $\{l_n^j\}$ are bounded from above and from below and there exist constant L_l^1, L_l^2 such that $\left\| S_n^{*^{-1}} e_{n-l} \right\| \leq L_l^1$, $\left\| Q_n^{*^{-1}} e_{-n+l} \right\| \leq L_l^2$ for $0 \leq l \leq M-1$. Then T is onto and hence invertible.*

The proof of Proposition 5.1 and Theorem 5.2 follow along similar lines as in the case of tridiagonal operator proved in [6].

Now our aim is to extend the verifiable criterion which was discussed in Sect. 4 to doubly infinite tridiagonal operator.

5.1 A verifiable criterion for a doubly infinite tridiagonal operators

Towards this end, let $V = \ell^2(\mathbb{Z})$. Let $\{e_n : n \in \mathbb{Z}\}$ be the standard orthonormal basis for V. Let T be a tridiagonal operator on V defined by

$$Te_n = u_{n-1}e_{n-1} + d_n e_n + l_{n+1}e_{n+1}.$$

Let

$$\begin{aligned} V_+ &= \overline{\operatorname{span}}\{e_n : n = 1, 2, \ldots\}, \\ V_- &= \overline{\operatorname{span}}\{e_n : n = -1, -2, \ldots\}, \\ V_0 &= \overline{\operatorname{span}}\{e_0\}. \end{aligned}$$

Then $V = V_- \bigoplus V_0 \bigoplus V_+$. Each vector $x = \sum_{i=-\infty}^{\infty} \alpha_i e_i \in V$ can be written uniquely as $x = x_- + x_0 e_0 + x_+$, where $x_- = \sum_{i=-\infty}^{-1} \alpha_i e_i \in V_-$, $x_+ = \sum_{i=1}^{\infty} \alpha_i e_i \in V_+$. Let P_- denote the orthogonal projection onto V_- and P_+ denote the orthogonal projection onto V_+. Then $x_- = P_-(x)$ and $x_+ = P_+(x)$. Let $S = P_+T_{|V_+}$, $Q = P_-T_{|V_-}$.

Theorem 5.3 *Suppose Q and S are invertible and satisfy the condition*

$$d_0 - l_0 u_{-1} \left\langle Q^{-1}e_{-1}, e_{-1} \right\rangle - u_0 l_1 \left\langle S^{-1}e_1, e_1 \right\rangle \neq 0.$$

Then the operator T is invertible and the equation $Tx = y$ can be solved uniquely for all $y \in V$.

Proof Let $y \in V$. The operator equation

$$Tx = y \tag{5.2}$$

can be written in a block matrix form as follows

$$\left[\begin{array}{ccccccc} & & & \vdots & & & \\ & Q & & 0 & & O & \\ & & & u_{-1} & & & \\ \cdots & 0 & l_0 & d_0 & u_0 & 0 & \cdots \\ & & & l_1 & & & \\ & O & & 0 & & S & \\ & & & \vdots & & & \end{array}\right] \begin{bmatrix} x_- \\ x_0 \\ x_+ \end{bmatrix} = \begin{bmatrix} y_- \\ y_0 \\ y_+ \end{bmatrix}.$$

In other words,

$$Qx_- + x_0 u_{-1} e_{-1} = y_-, \tag{5.3}$$
$$l_0 x_{-1} + d_0 x_0 + u_0 x_1 = y_0, \tag{5.4}$$
$$l_1 x_0 e_1 + S x_+ = y_+. \tag{5.5}$$

Given Q and S invertible. Then (5.3) and (5.5) imply

$$x_- = Q^{-1}(y_- - x_0 u_{-1} e_{-1}), \tag{5.6}$$
$$x_+ = S^{-1}(y_+ - l_1 x_0 e_1). \tag{5.7}$$

Note that

$$\begin{aligned} x_{-1} &= \langle x, e_{-1}\rangle = \langle x_-, e_{-1}\rangle, \\ &= \left\langle Q^{-1} y_-, e_{-1}\right\rangle - x_0 u_{-1}\left\langle Q^{-1} e_{-1}, e_{-1}\right\rangle. \end{aligned} \tag{5.8}$$

Similarly

$$\begin{aligned} x_1 &= \langle x, e_{-1}\rangle = \langle x_+, e_1\rangle, \\ &= \left\langle S^{-1} y_+, e_1\right\rangle - x_0 l_1\left\langle Q^{-1} e_1, e_1\right\rangle. \end{aligned} \tag{5.9}$$

Substituting these values of x_{-1} and x_1 in (5.4), we obtain

$$\begin{aligned} l_0\left\langle Q^{-1} y_-, e_{-1}\right\rangle - l_0 x_0 u_{-1}\left\langle Q^{-1} e_{-1}, e_{-1}\right\rangle + d_0 x_0 \\ + u_0\left\langle S^{-1} y_+, e_1\right\rangle - u_0 x_0 l_1\left\langle S^{-1} e_1, e_1\right\rangle = y_0. \end{aligned}$$

In other words,

$$\begin{aligned} x_0(-l_0 u_{-1}\left\langle Q^{-1} e_{-1}, e_{-1}\right\rangle + d_0 - u_0 l_1\left\langle S^{-1} e_1, e_1\right\rangle) \\ = (y_0 - l_0\left\langle Q^{-1} y_-, e_{-1}\right\rangle \\ - u_0\left\langle S^{-1} y_+, e_1\right\rangle). \end{aligned} \tag{5.10}$$

By the given hypothesis the value of x_0 can be obtained from (5.10). Using the value of x_0, the terms, x_- and x_+ can be obtained from (5.6) and (5.7). □

Example 5.1 Let $u_n = 1,\ n \neq -1, 0$ and $l_n = 1,\ n \neq 0, 1$ for all $n \in \mathbb{Z}$. For all $n \in \mathbb{Z} \setminus \{0\}$, let

$$d_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 10, & \text{if } n \text{ is even.} \end{cases}$$

Let $u_{-1} = l_1 = \frac{1}{4}$, $u_0 = l_0 = 1$ and $d_0 = 7$. Let $m_0 = \frac{1}{4}$. In this case, we have $S = Q$. It follows from Example 4.1 that S is invertible. And from the proof of Proposition 4.1, $\left\| S_n^{-1} e_n \right\| \leq \left(m_0 c + \frac{d}{1-m_0 uc} \right) \frac{1}{1-m_0 c^2}$ from which it turns out that $\left\| S^{-1} e_1 \right\| \leq 11$. The latter leads to

$$\begin{aligned}
&|d_0 - l_0 u_{-1} \left\langle Q^{-1} e_{-1}, e_{-1} \right\rangle - u_0 l_1 \left\langle S^{-1} e_1, e_1 \right\rangle | \\
&\geq |d_0| - |l_0 u_{-1}| |\left\langle Q^{-1} e_{-1}, e_{-1} \right\rangle | - |u_0 l_1| |\left\langle S^{-1} e_1, e_1 \right\rangle | \\
&\geq |d_0| - |l_0 u_{-1}| \left\| Q^{-1} e_{-1} \right\| - |u_0 l_1| \left\| S^{-1} e_1 \right\| \\
&= 7 - \frac{22}{4} > 0.
\end{aligned}$$

Thus the hypothesis of Theorem 5.3 is satisfied. Hence T is invertible.

5.2 A verifiable criterion for a doubly infinite penta diagonal operators and extensions

Now consider a bounded linear operator $T : V \to V$ defined by

$$T(e_n) = w_{n-2} e_{n-2} + u_{n-1} e_{n-1} + d_n e_n + l_{n+1} e_{n+1} + k_{n+2} e_{n+2}.$$

Consider the operator equation $Tx = y$ for some given $y \in V$. We can write

$$y = \sum_{n=-\infty}^{\infty} \beta_n e_n = y_- + \beta_0 y_0 + y_+.$$

Note that, for $i \leq -3$, $T(e_i) \in V_-$. Therefore

$$\begin{aligned}
Q(e_i) &= P_- T(e_i) = T(e_i), \\
T(e_{-2}) &= w_{-4} e_{-4} + u_{-3} e_{-3} + d_{-2} e_{-2} + l_{-1} e_{-1} + k_0 e_0 = Q(e_{-2}) + k_0 e_0, \\
T(e_{-1}) &= w_{-3} e_{-3} + u_{-2} e_{-2} + d_{-1} e_{-1} + l_0 e_0 + k_1 e_1 \\
&= Q(e_{-1}) + l_0 e_0 + k_1 e_1.
\end{aligned}$$

Since $x_- = \sum_{i \leq -3} \alpha_i e_i + \alpha_{-2} e_2 + \alpha_{-1} e_{-1}$, we have

$$T(x_-) = Q(x_-) + \alpha_{-2} k_0 e_0 + \alpha_{-1} l_0 e_0 + \alpha_{-1} k_1 e_1. \tag{5.11}$$

Furthermore,

$$T(e_0) = w_{-2} e_{-2} + u_{-1} e_{-1} + d_0 e_0 + l_1 e_1 + k_2 e_2. \tag{5.12}$$

Also, note that for $i \geq 3$, $T(e_i) \in V_+$. As a consequence

$$\begin{aligned}
S(e_i) &= P_+T(e_i) = T(e_i),\\
T(e_1) &= w_{-1}e_{-1} + u_0e_0 + d_1e_1 + l_2e_2 + k_3e_3 = w_{-1}e_{-1} + u_0e_0 + S(e_1),\\
T(e_2) &= w_0e_0 + u_1e_1 + d_2e_2 + l_3e_3 + k_4e_4 = w_0e_0 + S(e_2).
\end{aligned}$$

Thus

$$T(x_+) = S(x_+) + \alpha_1 w_{-1}e_{-1} + \alpha_1 u_0e_0 + \alpha_2 w_0e_0. \tag{5.13}$$

Making use of (5.11), (5.12), and (5.13), the equation $T(x) = y$ becomes

$$\begin{aligned}
T(x) = T(x_- + \alpha_0e_0 + x_+) &= T(x_-) + \alpha_0T(e_0) + T(x_+)\\
&= Q(x_-) + \alpha_{-2}k_0e_0 + \alpha_{-1}l_0e_0 + \alpha_{-1}k_1e_1\\
&+ \alpha_0w_{-2}e_{-2} + \alpha_0u_{-1}e_{-1} + \alpha_0d_0e_0 + \alpha_0l_1e_1 + \alpha_0k_2e_2\\
&+ S(x_+) + \alpha_1w_{-1}e_{-1} + \alpha_1u_0e_0 + \alpha_2w_0e_0\\
&= y_- + \beta_0e_0 + y_+.
\end{aligned}$$

Equating the components of V_-, V_0 and V_+ on both sides of the above equation, we get the following equations:

$$Q(x_-) + \alpha_0w_{-2}e_{-2} + \alpha_0u_{-1}e_{-1} + \alpha_1w_{-1}e_{-1} = y_-, \tag{5.14}$$

$$\alpha_{-2}k_0 + \alpha_{-1}l_0 + \alpha_0d_0 + \alpha_1u_0 + \alpha_2w_0 = \beta_0, \tag{5.15}$$

$$S(x_+) + \alpha_{-1}k_1e_1 + \alpha_0l_1e_1 + \alpha_0k_2e_2 = y_+. \tag{5.16}$$

Next we assume that Q and S are invertible. Then multiplying (5.14) by Q^{-1} and (5.16) by S^{-1}, we obtain the following equations,

$$x_- = Q^{-1}(y_-) - \alpha_0w_{-2}Q^{-1}(e_{-2}) - \alpha_0u_{-1}Q^{-1}(e_{-1}) - \alpha_1w_{-1}Q^{-1}(e_{-1}), \tag{5.17}$$

$$x_+ = S^{-1}(y_+) - \alpha_{-1}k_1S^{-1}(e_1) - \alpha_0l_1S^{-1}(e_1) - \alpha_0k_2S^{-1}(e_2). \tag{5.18}$$

Next note that

$$\begin{aligned}
\alpha_{-1} &= \langle x, e_{-1}\rangle = \langle x_-, e_{-1}\rangle\\
&= \langle Q^{-1}(y_-), e_{-1}\rangle - \alpha_0w_{-2}\langle Q^{-1}(e_{-2}), e_{-1}\rangle - \alpha_0u_{-1}\langle Q^{-1}(e_{-1}), e_{-1}\rangle\\
&- \alpha_1w_{-1}\langle Q^{-1}(e_{-1}), e_{-1}\rangle.
\end{aligned} \tag{5.19}$$

Similarly,

$$\begin{aligned}\alpha_{-2} &= \langle x, e_{-2}\rangle = \langle x_-, e_{-2}\rangle \\ &= \langle Q^{-1}(y_-), e_{-2}\rangle - \alpha_0 w_{-2}\langle Q^{-1}(e_{-2}), e_{-2}\rangle - \alpha_0 u_{-1}\langle Q^{-1}(e_{-1}), e_{-2}\rangle \\ &- \alpha_1 w_{-1}\langle Q^{-1}(e_{-1}), e_{-2}\rangle. \end{aligned} \tag{5.20}$$

Also,

$$\begin{aligned}\alpha_1 &= \langle x, e_1\rangle = \langle x_+, e_1\rangle \\ &= \langle S^{-1}(y_+), e_1\rangle - \alpha_{-1}k_1\langle S^{-1}(e_1), e_1\rangle - \alpha_0 l_1\langle S^{-1}(e_1), e_1\rangle \\ &- \alpha_0 k_2\langle S^{-1}(e_2), e_1\rangle. \end{aligned} \tag{5.21}$$

$$\begin{aligned}\alpha_2 &= \langle x, e_2\rangle = \langle x_+, e_2\rangle \\ &= \langle S^{-1}(y_+), e_2\rangle - \alpha_{-1}k_1\langle S^{-1}(e_1), e_2\rangle - \alpha_0 l_1\langle S^{-1}(e_1), e_2\rangle \\ &- \alpha_0 k_2\langle S^{-1}(e_2), e_2\rangle. \end{aligned} \tag{5.22}$$

Equations (5.19), (5.20), (5.21), and (5.22) along with (5.15) form a system of 5 equations in 5 unknowns, namely, $\alpha_i, \ -2 \le i \le 2$. Let $\gamma_{ij} = \langle Q^{-1}(e_{-j}), e_{-i}\rangle, \quad \delta_{ij} = \langle S^{-1}(e_j), e_i\rangle$. Then we can rewrite the above mentioned system of equation in simplified form as,

$$k_0\alpha_{-2} + l_0\alpha_{-1} + d_0\alpha_0 + u_0\alpha_1 + w_0\alpha_2 = \beta_0, \tag{5.23}$$

$$\alpha_{-1} + (w_{-2}\gamma_{12} + u_{-1}\gamma_{11})\alpha_0 + w_{-1}\gamma_{11}\alpha_1 = \langle Q^{-1}(y_-), e_{-1}\rangle, \tag{5.24}$$

$$\alpha_{-2} + (w_{-2}\gamma_{22} + u_{-1}\gamma_{21})\alpha_0 + w_{-1}\gamma_{21}\alpha_1 = \langle Q^{-1}(y_-), e_{-2}\rangle, \tag{5.25}$$

$$\alpha_1 + k_1\delta_{11}\alpha_{-1} + (l_1\delta_{11} + k_2\delta_{12})\alpha_0 = \langle S^{-1}(y_+), e_1\rangle, \tag{5.26}$$

$$\alpha_2 + k_1\delta_{21}\alpha_{-1} + (l_1\delta_{21} + k_2\delta_{22})\alpha_0 = \langle S^{-1}(y_+), e_2\rangle. \tag{5.27}$$

Now let $\mathcal{M}$ be the matrix of coefficients of the above system of equations (5.23), (5.24), (5.25), (5.26), and (5.27). Note that $\mathcal{M}$ is of order 5×5. Suppose $\mathcal{M}$ is invertible. Then once y is known, the right hand side of all equations is known.(Recall that we have assumed that Q and S are invertible.) Hence this system has a unique solution $\alpha_i, \ -2 \le i \le 2$. Once these values are known, then x_- and x_+ can be determined uniquely from equations (5.17) and (5.18), respectively. Hence x is uniquely determined. In other words, T is invertible.

Theorem 5.4 *Suppose that Q, S are invertible and that the matrix $\mathcal{M}$ defined as above is also invertible. Then the operator T is invertible and the equation $Tx = y$ can be solved uniquely for all $y \in V$.*

Remark 5.5 The above theorem and its proof can be extended to a $(2M + 1)$-diagonal operator. The matrix $\mathcal{M}$ in that case will be of the order $(2M + 1) \times (2M + 1)$. The calculations to obtain that matrix $\mathcal{M}$ may be lengthy and involved.

6 An application to a sampling problem

In modern digital data processing, mathematical sampling theory plays an important role. Mathematical sampling theory deals with sampling and reconstruction of a signal (or an image) from its sample points. The signals formally belong to an appropriate subclass of $L^2(\mathbb{R})$. We refer to the work of Butzer and Stens [14] for the classical historical review of sampling theory. After the foundation of the theory of wavelets and multi resolution analysis [26, 27], the sampling problems have been considered in a subclass of $L^2(\mathbb{R})$, namely shift-invariant spaces. We refer to some papers in the literature which involve sampling and reconstruction in a shift-invariant space see e.g. [1–5, 19, 20, 25, 28, 29, 33, 34].

Let $f \in L^1(\mathbb{R})$. Then the Fourier transform $\widehat{f}$ of f is defined by

$$\widehat{f}(\xi) := \int\limits_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \mathrm{d}x \quad a.e.\ \xi \in \mathbb{R}.$$

Then $\widehat{f} \in C_0(\mathbb{R})$, the class of continuous functions vanishing at ∞. Further, if $f \in L^1(\mathbb{R})$, and $\widehat{f} \in L^1(\mathbb{R})$, then the following inversion formula is valid

$$f(x) = \int\limits_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi i x \xi} \mathrm{d}\xi \quad a.e.\ x \in \mathbb{R}.$$

The Fourier transform initially defined for $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ can be extended to an isometric isomorphism of $L^2(\mathbb{R})$ onto itself.

Definition 6.1 A closed subspace E of $L^2(\mathbb{R})$ is called a shift-invariant space if $T_k\phi \in E$, for every $\phi \in E$ and $k \in \mathbb{Z}$, where T_k is a translation operator defined as $T_k\phi(x) := \phi(x-k)$ for all $x \in \mathbb{R}$. For $\phi \in L^2(\mathbb{R})$, $\overline{\text{span}}\{T_k\phi : k \in \mathbb{Z}\}$ is called the shift-invariant space generated by ϕ and is denoted by $V(\phi)$.

Definition 6.2 A Riesz basis for a separable Hilbert space $\mathcal{H}$ is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded invertible operator.

Theorem 6.3 *Let $\phi \in Ł^2(\mathbb{R})$. Then $\{T_k\phi : k \in \mathbb{Z}\}$ is a Riesz basis for $V(\phi)$ if and only if there exist $A, B > 0$ such that*

$$0 < A \leq G_\phi(\xi) \leq B < \infty \; a.e.\ \xi \in \mathbb{R},$$

where $G_\phi(\xi) = \sum\limits_{k \in \mathbb{Z}} |\widehat{\phi}(\xi + k)|^2$.

For more details we refer to [15].

Definition 6.4 The Wiener amalgam space $W(C, \ell^1)$ is defined as

$$W(C, \ell^1) := \left\{ f \in C(\mathbb{R}) : \sum_{n \in \mathbb{Z}} \max_{x \in [0,1]} |f(x+n)| < \infty \right\}.$$

A closed subspace E of $L^2(\mathbb{R})$ is said to be a reproducing kernel Hilbert space if for all $x \in \mathbb{R}$ there exists a function $K_x \in E$ such that

$$f(x) = \langle f, K_x \rangle \ \forall\, f \in E.$$

The function $K(x, y) = K_x(y) = \left\langle K_x, K_y \right\rangle$ is called the reproducing kernel of E.

If $\phi \in W(C, \ell^1)$ such that $\{T_k\phi : k \in \mathbb{Z}\}$ is a Riesz basis for $V(\phi)$ then each function in $V(\phi)$ is continuous and $V(\phi)$ is a reproducing kernel Hilbert space. In particular if ϕ is a continuous function with compact support, then $V(\phi)$ is a reproducing kernel Hilbert space. The reproducing kernel is given by

$$K(x, y) = \sum_{k\in\mathbb{Z}} \overline{\phi(x-k)}\tilde{\phi}(y-k)$$

where $\{T_k\tilde{\phi} : k \in \mathbb{Z}\}$ is the dual Riesz basis of $\{T_k\phi : k \in \mathbb{Z}\}$.

Definition 6.5 A set $X = \{x_n : n \in \mathbb{Z}\}$ is said to be a stable set of sampling for a closed subspace V of $L^2(\mathbb{R})$ if there exist constants $0 < m \leq M < \infty$ such that

$$m\|f\|_2 \leq \left(\sum_{n\in\mathbb{Z}} |f(x_n)|^2\right)^{1/2} \leq M\|f\|_2$$

for every $f \in V$.

Let U be the infinite matrix with entries $U_{ij} = \phi(x_i - j),\ i, j \in \mathbb{Z}$. Then it is well known in the literature that $X = \{x_j : j \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ if and only if there exist $A, B > 0$ such that

$$A\|c\|^2_{\ell^2(\mathbb{Z})} \leq \|Uc\|^2_{\ell^2(\mathbb{Z})} \leq B\|c\|^2_{\ell^2(\mathbb{Z})}$$

for every $c \in \ell^2(\mathbb{Z})$.

The study of invertibility of an infinite band matrix leads to the following application.

Theorem 6.6 *Let ϕ be a continuous function having support in [-1,1] such that $\{T_k\phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$. Let $X = \{x_j : j \in \mathbb{Z}\}$. Let $U = [U_{ij}]_{i,j\in\mathbb{Z}} = [\phi(x_i - j)]_{i,j\in\mathbb{Z}}$ be an infinite matrix associated with ϕ. Suppose U^*U has the form*

$$\begin{bmatrix} & & \vdots & & \\ & Q & 0 & O & \\ & & U_{-10} & & \\ \cdots 0 & U_{0-1} & U_{00} & U_{01} & 0 \cdots \\ & & U_{10} & & \\ & O & 0 & S & \\ & & \vdots & & \end{bmatrix},$$

where Q and S satisfy the hypothesis of Theorem 5.3. Then X is a stable set of sampling for $V(\phi)$.

Proof Since $\text{supp}(\phi) \subset [-1, 1]$, it can be easily shown that U^*U is tridiagonal. Then it follows from Theorem 5.3 that U^*U is invertible. This means U is bounded above and below from which it follows that X is a stable set of sampling for $V(\phi)$. □

Remark 6.7 For $\text{supp}(\phi) \subset [-M, M]$, it can be easily shown that U^*U is $(2M+1)$ diagonal. Then it follows from Theorem 5.4 and Remark 5.5 that X is a stable set of sampling for $V(\phi)$.

We shall illustrate the Theorem 6.6 with the following

Example 6.1 Define

$$\phi(x) = \begin{cases} -99x + 34, & x \in [0, 1/3], \\ 3x, & x \in [1/3, 2/3], \\ -6x + 6, & x \in [2/3, 1], \\ 0 & x \geq 1. \end{cases} \tag{6.1}$$

Extend ϕ on $[-1, 0]$ as $\phi(x) = \phi(-x)$. Then $\text{supp}\,\phi \subseteq [-1, 1]$ with $\phi(\pm 1) = 0$. Now we want to show that $\{T_k\phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$. By Poission summation formula, $G_\phi(\xi) = c_0 + 2\sum_{k=1}^{\infty} c_k \cos(2\pi k\xi)$, where $c_k = \int_{-\infty}^{\infty} \phi(x)\phi(x+k)\mathrm{d}x$. Then $c_0 = \int_{-1}^{1}(\phi(x))^2\mathrm{d}x = 267.11$, $c_1 = \int_{-1}^{1}\phi(x)\phi(x+1)\mathrm{d}x = 8.72$. Therefore $G_\phi(\xi) = 267.11 + 17.44\cos(2\pi\xi)$. Clearly $249 \leq G_\phi(\xi) \leq 285$. Hence, from Theorem 6.3, it follows that $\{T_k\phi : k \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$.

Let $X = \{\frac{2}{3}j : j \in \mathbb{Z}\}$. Then $U_{i,j} = \phi(\frac{2}{3}i - j)$. Using straightforward computations, we obtain

$$U^*U = \begin{bmatrix} \ddots & \ddots & \ddots & & & & & \\ & 0 & 2 & 2 & 2 & 0 & & \\ & & 0 & 2 & 1164 & 2 & 0 & \\ & & & 0 & 2 & 2 & 2 & 0 \\ & & & & \ddots & & \ddots & \ddots \end{bmatrix}.$$

We note that U^*U is tridiagonal and it is not a diagonally dominant matrix. We consider the matrix S in U^*U with $m_0 = \frac{1}{9}$. The entries of S satisfy the hypothesis of Theorem 4.2. Hence S is invertible. Further $(U^*U)_{-i,-j} = (U^*U)_{i,j}\ \forall\, i, j \in \mathbb{N}$. In other words $S = Q$. We also have $(U^*U)_{00} = 1164$, $(U^*U)_{10} = (U^*U)_{-10} = (U^*U)_{01} = (U^*U)_{0-1} = 2$ and $(U^*U)_{0j} = (U^*U)_{i0} = 0$ for all i, j. This satisfies the hypothesis of Theorem 5.3. Thus U^*U is invertible and X is a stable set of sampling for $V(\phi)$.

Acknowledgements The authors thank the referees for meticulously reading the manuscript and providing valuable comments and suggestions. Some part of the work was done when the second author was visiting the Center of Mathematics, Technical University Munich during October 2019 to September 2020. She sincerely thanks Professor Massimo Fornasier and Professor Peter Massopust, TUM, for their kind invitation and excellent hospitality for the entire period of visit.

References

1. Acosta-Reyes, E., Aldroubi, A., Krishtal, I.: On stability of sampling-reconstruction models. Adv. Comput. Math. **31**(1–3), 5–34 (2009)
2. Aldroubi, A., Gröchenig, K.: Nonuniform sampling and reconstruction in shift-invariant spaces. SIAM Rev. **43**(4), 585–620 (2001)
3. Aldroubi, A., Krishtal, I.: Robustness of sampling and reconstruction and Beurling–Landau-type theorems for shift-invariant spaces. Appl. Comput. Harmon. Anal. **20**(2), 250–260 (2006)
4. Aldroubi, A., Sun, Q., Tang, W.S.: Nonuniform average sampling and reconstruction in multiply generated shift-invariant spaces. Constr. Approx. **20**(2), 173–189 (2004)
5. Antony Selvan, A., Radha, R.: Sampling and reconstruction in shift-invariant spaces on $\mathbb{R}^d$. Ann. Mat. Pura Appl. (4) **194**(6), 1683–1706 (2015)
6. Antony Selvan, A., Radha, R.: Invertibility of a tridiagonal operator with an application to a non-uniform sampling problem. Linear Multilinear Algebra **65**(5), 973–990 (2017)
7. Arveson, W.: C^*-algebras and numerical linear algebra. J. Funct. Anal. **122**(2), 333–360 (1994)
8. Arveson, W.: The role of C^*-algebras in infinite-dimensional numerical linear algebra. In: C^*-Algebras: 1943–1993 (San Antonio, TX, 1993), vol. 167 of Contemp. Math., pp. 114–129. Amer. Math. Soc., Providence, RI (1994)
9. Axelsson, O., Barker, V.A.: Finite Element Solution of Boundary Value Problems: Theory and Computation. Computer Science and Applied Mathematics, Academic Press Inc., Orlando, FL (1984)
10. Balasubramanian, R., Kulkarni, S.H., Radha, R.: Solution of a tridiagonal operator equation. Linear Algebra Appl. **414**(1), 389–405 (2006)
11. Böttcher, A., Silbermann, B.: The finite section method for Toeplitz operators on the quarter-plane with piecewise continuous symbols. Math. Nachr. **110**, 279–291 (1983)
12. Böttcher, A., Silbermann, B.: Analysis of Toeplitz Operators. Springer-Verlag, Berlin (1990)
13. Burden, R.L., Faires, J.D.: Numerical Analysis. Brooks/Cole, Cengage Learning, Boston, MA (2005)
14. Butzer, P.L., Stens, R.L.: Sampling theory for not necessarily band-limited functions: a historical overview. SIAM Rev. **34**(1), 40–53 (1992)
15. Christensen, O.: Frames and Bases. An Introductory Course. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc, Boston, MA (2008)
16. Christensen, O., Strohmer, T.: The finite section method and problems in frame theory. J. Approx. Theory **133**(2), 221–237 (2005)
17. Donatelli, M., Garoni, C., Carla, M., Manni, C., Serra-Capizzano, S., Speleers, H.: Symbol-based multigrid methods for Galerkin b-spline isogeometric analysis. SIAM J. Numer. Anal. **55**(1), 31–62 (2017)
18. Elsgolts, L.: Differential Equations and the Calculus of Variations. Translated from the Russian by George Yankovsky, Mir Publishers, Moscow (1977)
19. Führ, H., Xian, J.: Relevant sampling in finitely generated shift-invariant spaces. J. Approx. Theory (2018). https://doi.org/10.1016/j.jat.2018.09.009
20. García, A.G., Pérez-Villalón, G., Portal, A.: Riesz bases in $L^2(0, 1)$ related to sampling in shift-invariant spaces. J. Math. Anal. Appl. **308**(2), 703–713 (2005)
21. Garoni, C., Serra-Capizzano, S., Sesana, D.: Spectral analysis and spectral symbol of d-variate Q_p Lagrangian FEM stiffness matrices. SIAM J. Matrix Anal. Appl. **36**(3), 1100–1128 (2015)
22. Gelfand, I.M., Fomin, S.V.: Calculus of Variations. Revised English Edition Translated and Edited by Richard A. Silverman. Prentice-Hall Inc., Englewood Cliffs, NJ (1963)
23. Gohberg, I.C., Fel'dman, I.A.: Convolution equations and projection methods for their solution. American Mathematical Society, Providence, RI (1974) (translated from the Russian by F. M. Goldware, translations of Mathematical Monographs, vol. 41)
24. Gohberg, I., Kaashoek, M.A., Schagen, F.V.: Finite section method for difference equations. In: Linear Operators and Matrices, vol. 130 Operator Theory: Advances and Applications Book Series, pp. 197–207. Birkhäuser, Basel (2002)
25. Liu, Y.M., Walter, G.G.: Irregular sampling in wavelet subspaces. J. Fourier Anal. Appl. **2**(2), 181–189 (1995)
26. Mallat, S.G.: Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$. Trans. Am. Math. Soc. **315**(1), 69–87 (1989)
27. Meyer, Y.: Ondelettes et fonctions splines. In: Séminaire sur les équations aux dérivées partielles 1986–1987, pp. Exp. No. VI, 18. École Polytech., Palaiseau (1987)

28. Nashed, M.Z., Sun, Q.: Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$. J. Funct. Anal. **258**(7), 2422–2452 (2010)
29. Radha, R., Sarvesh, K., Sivananthan, S.: Sampling and reconstruction in a shift invariant space with multiple generators. Numer. Funct. Anal. Optim. (2018). https://doi.org/10.1080/01630563.2018.1501701
30. Robert, L., Santiago, L.: Finite sections method for Hessenberg matrices. J. Approx. Theory **123**(1), 68–88 (2003)
31. Serra Capizzano, S., Tablino Possio, C.: Spectral and structural analysis of high precision finite difference matrices for elliptic operators. Linear Algebra Appl. **293**(1–3), 85–131 (1999)
32. Strikewerda, J.C.: Finite Difference Schemes and Partial Differential Equations. International Thompson Publ., Chapman and Hall, New York (1989)
33. Sun, W., Zhou, X.: Average sampling in spline subspaces. Appl. Math. Lett. **15**(2), 233–237 (2002)
34. Sun, Q.: Nonuniform average sampling and reconstruction of signals with finite rate of innovation. SIAM J. Math. Anal. **38**(5):1389–1422 (2006/07)