$$g'(z) = g(z)\sum_{k=0}^{p-1}\omega^k\cot(\omega^k z) = \left(z^p - \frac{\zeta(2p)}{\pi^{2p}}z^{3p} + \cdots\right)\sum_{k=0}^{p-1}\omega^k\cot(\omega^k z). \tag{5}$$

To evaluate $\zeta(2p)$ we equate the coefficient of z^{3p-1} in (4) with that in (5). Contributions to this coefficient in (5) come from two sources arising from the Laurent expansion of the sum of cotangents, namely, from the coefficient of z^{-1} and from the coefficient of z^{2p-1}. Because

$$\cot z = \frac{1}{z} + \sum_{r=1}^{\infty} c_r z^{2r-1},$$

where $c_r = (-1)^r 2^{2r} B_{2r}/(2r)!$,

$$\omega^k\cot(\omega^k z) = \frac{1}{z} + \omega^k\sum_{r=1}^{\infty} c_r(\omega^k z)^{2r-1} = \frac{1}{z} + \sum_{r=1}^{\infty} c_r\omega^{2rk}z^{2r-1}.$$

When this is summed over k the total contribution from z^{-1} is p, while that from z^{2p-1} is pc_p, because $\omega^{2kp} = 1$. Equating the coefficient of z^{3p-1} in (4) with the corresponding one in (5), we find that

$$-3p\frac{\zeta(2p)}{\pi^{2p}} = -p\frac{\zeta(2p)}{\pi^{2p}} + pc_p.$$

This gives $\zeta(2p) = -c_p\pi^{2p}/2 = (-1)^{p+1}2^{2p-1}B_{2p}/(2p)!$, as required. ■

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A Very Simple and Elementary Proof of a Theorem of Ingelstam

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1. INTRODUCTION. The aim of this note is to give a very simple and elementary proof of the following interesting theorem due to Ingelstam [**4**].

Theorem 1.1. *Let A be a real algebra with unit* 1*. Suppose that A is also a real Hilbert space such that* $\|1\| = 1$ *and* $\|ab\| \leq \|a\|\|b\|$ *for all a and b in A. Then A is isomorphic to the algebra* $\mathbb{R}$ *of real numbers, the algebra* $\mathbb{C}$ *of complex numbers, or the algebra* $\mathbb{H}$ *of real quaternions.*

The words "very simple and elementary" in the title mean that any undergraduate student who has taken a decent linear algebra course should be able to understand this proof. Though the statement of the theorem contains the words "Hilbert space," we do not assume any familiarity with Hilbert spaces. We assume only some well-known properties of inner product spaces that are stated explicitly in the next section. We also replace some of the assumptions in the theorem by weaker assumptions. (See Theorem 3.1 for the precise statement.)

Ingelstam's proof of Theorem 1.1 used techniques from Banach algebras, specifically the vertex property for Banach algebras. Subsequently, progressively simpler proofs were given by Smiley [**6**], Froelich [**3**], and Zalar [**7**]. However all these proofs also made use of Banach algebra techniques. The proofs of Smiley and Froelich used the Gelfand theory, whereas the proof of Zalar used the famous theorem due to Gelfand, Mazur, and Arens asserting that every normed division algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. As mentioned earlier, we do not use any of these results. Our proof applies to the case of a real associative algebra with unit. Zalar's paper [**7**] also contains proofs for the case of algebras without unit. The proofs are not elementary and use ideas related to topological divisors of zero. It would be interesting to know if an elementary proof can be found for this case as well. In the same paper, Zalar also considered the case of nonassociative algebras.

2. PRELIMINARIES.

We assume that the reader has knowledge of the following: *real vector space*, *real inner product space*, and *ring*. We use the following standard facts about a real inner product space.

Let $\langle , \rangle$ denote the inner product on an inner product space V. For an arbitrary subset S of V, we denote by $S^\perp$ the *orthogonal complement* of S. Thus

$$S^\perp := \big\{x \in V : \langle x, s\rangle = 0 \text{ for all } s \in S\big\}.$$

The inner product $\langle , \rangle$ induces a *norm* on V given by $\|a\| = \langle a, a\rangle^{1/2}$ for a in V.

Fact 1. Let S be a finite subset of an inner product space V. Then $S^\perp = \{0\}$ if and only if $\operatorname{span}(S) = V$. This follows by observing that if $S = \{a_1, \ldots, a_n\}$ and x is in V, then $x - \sum_{j=1}^{n} \langle x, a_j\rangle a_j$ belongs to $S^\perp$.

Fact 2 (Cauchy- Schwarz inequality). For all x and y in V,

$$\big|\langle x, y\rangle\big| \leq \|x\|\|y\|,$$

and equality holds if and only if $\{x, y\}$ is a linearly dependent set.

A *real algebra A* is a real vector space on which a binary operation of multiplication is defined that makes A into a ring (with respect to addition and multiplication of vectors) and also satisfies $(\alpha a)b = \alpha(ab) = a(\alpha b)$ for all a, b in A and α in $\mathbb{R}$.

Examples of real algebras include the field $\mathbb{R}$ of real numbers (with the usual operations), the field $\mathbb{C}$ of complex numbers, and the division ring $\mathbb{H}$ of real quaternions. The last of these algebras is constructed by defining a multiplication on the real vector space $\mathbb{R}^4$ as follows. Let $1 = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$, and

$k = (0, 0, 0, 1)$ denote the usual standard basis vectors of $\mathbb{R}^4$. Define

$$1^2 = 1,\ 1i = i = i1,\ 1j = j = j1,\ 1k = k = k1,$$
$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

and extend this multiplication to $\mathbb{R}^4$ by linearity.

The essence of our main theorem is that under certain conditions these three are the only examples of real algebras. For other examples of algebras and also for the theory of Banach algebras, the interested reader may refer to [**2**] or [**5**].

3. MAIN THEOREM. As in [**7**], we replace the hypotheses in Ingelstam's theorem (Theorem 1.1) by the weaker assumptions:

(1) A is a real inner product space (instead of a real Hilbert space).

(2) The inequality $\|a^2\| \leq \|a\|^2$ holds for all a in A (in place of the original assumption that $\|ab\| \leq \|a\|\|b\|$ for all a and b in A).

Theorem 3.1. *Let A be a real algebra with unit* 1. *Suppose that A is also a real inner product space such that $\|1\| = 1$ and $\|a^2\| \leq \|a\|^2$ for all a in A. Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

Proof. We first establish the following two claims.

Claim 1. *If x belongs to $\{1\}^\perp$ and $\|x\| = 1$, then $x^2 = -1$.*

We argue as in [**7**]. For each t in $\mathbb{R}$, we have

$$(t^2 + 1)^2 = \left(\|t + x\|^2\right)^2 \geq \left\|(t + x)^2\right\|^2 = \langle t^2 + 2tx + x^2, t^2 + 2tx + x^2\rangle.$$

This implies that

$$2t^2\left(1 + \langle 1, x^2\rangle\right) + 4t\langle x, x^2\rangle + \|x^2\|^2 - 1 \leq 0.$$

Since this holds for all real t, we have

$$1 + \langle 1, x^2\rangle \leq 0$$

that is

$$\langle 1, x^2\rangle \leq -1.$$

The Cauchy-Schwartz inequality implies that $\langle 1, x^2\rangle = -1$. The condition for equality in the Cauchy-Schwartz inequality ensures that $x^2 = -1$.

Claim 2. *If x and y are in $\{1\}^\perp$, $\langle x, y\rangle = 0$, and $\|x\| = 1 = \|y\|$, then $yx = -xy$.*

Since x and y belong to $\{1\}^\perp$, we have $(x + y)/\sqrt{2}$ in $\{1\}^\perp$. Also, since $\langle x, y\rangle = 0$ and $\|x\| = 1 = \|y\|$, we obtain $\|(x + y)/\sqrt{2}\| = 1$. Hence, by Claim 1, $x^2 = -1 = y^2$ and $((x + y)/\sqrt{2})^2 = -1$. This implies that $xy + yx = 0$.

We now move to the proof of the theorem. If $\{1\}^\perp = \{0\}$, then by Fact 1 we have $A = \text{span}\{1\}$, which is isomorphic to $\mathbb{R}$.

Now suppose that $\{1\}^\perp \neq \{0\}$. Then there exists x in $\{1\}^\perp$ with $\|x\| = 1$. By Claim 1, $x^2 = -1$. If $A = \text{span}\{1, x\}$, then A is obviously isomorphic to $\mathbb{C}$. Otherwise there exists y in $\{1, x\}^\perp$ with $\|y\| = 1$. By Claim 1, $y^2 = -1$, and by Claim 2, $yx = -xy$. Consider $z = xy$. Then

$$z^2 = xyxy = -yxxy = -y(-1)y = y^2 = -1.$$

Also,

$$yz = yxy = -xy^2 = x, \qquad zy = xy^2 = -x,$$

$$zx = xyx = -yx^2 = y, \qquad xz = xxy = -y.$$

Thus we have shown that $x^2 = y^2 = z^2 = -1$, $xy = -yx = z$, $yz = -zy = x$, and $zx = -xz = y$. These relations also imply that $\{1, x, y, z\}$ is a linearly independent set. For if $u = \alpha + \beta x + \gamma y + \delta z = 0$, consider $v = \alpha - \beta x - \gamma y - \delta z$. Then $0 = uv = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$, that is, $\alpha = \beta = \gamma = \delta = 0$. Thus $\text{span}\{1, x, y, z\}$ is isomorphic to $\mathbb{H}$. Now if $A = \text{span}\{1, x, y, z\}$, then A is isomorphic to $\mathbb{H}$. If not, there exists u in $\{1, x, y, z\}^\perp$ with $\|u\| = 1$. By Claim 1, $u^2 = -1$, and by Claim 2, $xu = -ux$, $yu = -uy$, and $zu = -uz$. But then

$$uz = u(xy) = (ux)y = (-xu)y = -x(uy) = -x(-yu) = (xy)u = zu = -uz.$$

Hence $uz = 0$. On the other hand, since $uz = zu$, we also have $(uz)^2 = u^2z^2 = (-1)(-1) = 1$. This is clearly impossible. (This last part of the proof follows the lines of the proof of the Gelfand-Mazur-Arens theorem as given in [**2**].) ■

Remark. Froelich's proof [**3**] contains a small (fixable) gap. In the proof, Froelich shows that A/M is a field (in fact, a real commutative normed division algebra) and concludes that A/M is isomorphic to $\mathbb{R}$. This is incorrect, for a real commutative normed division algebra can be isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. If A/M is isomorphic to $\mathbb{R}$, Froelich's proof works. If A/M is isomorphic to $\mathbb{C}$, then the composition of this isomorphism with the quotient map of A onto A/M is a (real algebra) homomorphism ϕ of A onto $\mathbb{C}$. Now define $q : A \to \mathbb{R}$ by $q(x) = \text{Re}\,\phi(x)$ for x in A. Then q is a continuous linear functional on A, with $\|q\| = 1 = q(1)$. The remaining part of the proof is the same as that given in [**3**].

In [**1**], the following analogue of Theorem 3.1 for real *-algebras was proved. Since the proof presented in [**1**] was based on Theorem 3.1, it can now be regarded as elementary. We include the theorem and a sketch of its proof for the sake of completeness. We recall that a *real *-algebra* is a real algebra A with a mapping $a \to a^*$ of A into A that satisfies the following axioms:

(1) $(a + b)^* = a^* + b^*$ for all a and b in A;

(2) $(\alpha a)^* = \alpha a^*$ for all a in A and α in $\mathbb{R}$;

(3) $(ab)^* = b^*a^*$ for all a and b in A;

(4) $(a^*)^* = a$ for all a in A.

Examples of real *-algebras include the field $\mathbb{R}$ of real numbers (with $a^* = a$ for all a), the field $\mathbb{C}$ of complex numbers (with a^* defined to be the complex conjugate of a), and the division ring $\mathbb{H}$ of real quaternions, where for $a = \alpha + \beta i + \gamma j + \delta k$ in $\mathbb{H}$, a^* is defined by $a^* = \alpha - \beta i - \gamma j - \delta k$. For other examples of *-algebras and also for the theory of Banach *-algebras , the interested reader may refer to [**2**] or [**5**].

Theorem 3.2. *Let A be a real *-algebra with unit* 1 *satisfying the following condition: for a in A,* $a^*a = 0$ *implies that* $a = 0$*. Suppose that A is also a real inner product space such that* $\|1\| = 1$ *and* $\|a^*a\| \leq \|a\|^2$ *for all a in A. Then A is isomorphic to* $\mathbb{R}$*,* $\mathbb{C}$*, or* $\mathbb{H}$*.*

Proof. Let $\mathrm{Sym}(A) := \{a \in A : a^* = a\}$, and let a belong to $\mathrm{Sym}(A)$. Consider the subalgebra B of A that comprises all polynomials in a with real coefficients. Then B is contained in $\mathrm{Sym}(A)$. Hence B satisfies hypotheses of Theorem 3.1. Since B is also commutative, B is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Hence $a = \lambda 1$ for some real or complex number λ. This shows that $\mathrm{Sym}(A)$ itself is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Now the conclusion follows from Lemma 2.1 of [**1**]. ■

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The Early History of the Ham Sandwich Theorem

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The following theorem is the well-known ham sandwich theorem: *for any three given sets in Euclidean space, each of finite outer Lebesgue measure, there exists a plane that bisects all three sets, i.e., separates each of the given sets into two sets of equal measure.* The early history of this result seems not to be well known. Stone and Tukey [**2**] attribute the theorem to Ulam. They say they got the information from a referee. Is this correct? The problem appears in *The Scottish Book* [**1**] as problem 123. The problem is posed by Steinhaus. A reference is made to the pre-World War II journal *Mathesis Polska* (Latin for "Polish Mathematics"). This journal is not easy to locate. It was finally located in the mathematics library of the University of Illinois, which seems to be the only library in the United States having the complete journal. One of the items