WIENER'S THEOREM, INFINITE MATRICES AND BANACH ALGEBRAS

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1. Introduction

This article is based on an invited talk given by me at the National Symposium on Mathematical Methods and Applications(NSMAA 2009) organized by the the Department of Mathematics, Indian Institute of Technology Madras on December 22, 2009. The Department has been organizing such a symposium every year in honour of the celebrated Indian Mathematician Srinivasa Ramanujan. I thank the organizers for inviting me to give a talk in this symposium. It is an honour to be associated with the illustrious memeory of Ramanujan.

An objecive of this talk is to highlight the connections between some apparently unrelated theorems and the role of Banach algebras in these theorems. The first of these theorems is the following well known theorem due to the famous mathematician Norbert Wiener. Wiener's proof can be found in his book [11].

Theorem 1.1. Wiener's Theorem: *Let f be a periodic function on $[-\pi, \pi]$. Suppose f has an absolutely convergent Fourier series and $f(t) \neq 0$ for all $t \in [-\pi, \pi]$. Then $1/f$ also has absolutely convergent Fourier series.*

Gelfand gave an elegant proof of this theorem using the techniques from Banach Algebras in his celebrated paper [2]. This was the first paper in which the theory of Banach algebras was developed systematically. Gelfand's proof is much shorter than the original proof of Wiener and it attracted the attention of Mathematicians to the theory of Banach algebras.

The second theorem is due to Jaffard [5] and deals with the decay of off-diagonal entries of doubly infinite matrices. Note that such matrices can be regarded as the operators on $\ell^2(\mathbb{Z})$, the space of all square summable doubly infinite sequences of complex numbers.

Theorem 1.2. Jaffard's theorem: *If a matrix A with entries $A(k,l), k,l \in \mathbb{Z}$ is invertible on $\ell^2(\mathbb{Z})$ and there are constants $C > 0$, $r > 1$ such that*

$$|A(k,l)| \leq C(1+|k-l|)^{-r} \text{for all} \quad k,l \in \mathbb{Z},$$

then,

$$|A^{-1}(k,l)| \leq C(1+|k-l|)^{-r} \text{for all} \quad k,l \in \mathbb{Z}.$$

In order to discuss next theorem in our list, we need a definition.

Definition 1.3. *Band dominated operators*
We shall regard an infinite matrix $A = [A(k,l)]$, $k,l \in \mathbb{Z}$ as an operator on $\ell^2(\mathbb{Z})$. Such a matrix is called a *band matrix* or *band operator* if there exists $n \in \mathbb{N}$ such that $A(k,l) = 0$ for $|k-l| > n$. A *band dominated matrix (or operator)* is a limit (in the operator norm) of a sequence of band operators.

Then the theorem can be simply stated as follows: If a band domonated operator is invertible, then its inverse is also band dominated. (Note that this not true for band operators.)
A more detailed statement is given below.

Theorem 1.4. Theorem: *For an infinite matrix $A = [A(k,l)]$, $k,l \in \mathbb{Z}$, let A_n denote the nth band approximation of A given by $A_n(k,l) = A(k,l)$ for $|k-l| \le n$ and $A_n(k,l) = 0$ otherwise. If A is invertible on $\ell^2(\mathbb{Z})$ and there exist positive constants r, C such that*

$$\|A - A_n\| \le Cn^{-r} \textit{for all} \quad n \in \mathbb{N},$$

then there exists a sequence $\{B_n\}$ of band matrices such that

$$\|A^{-1} - B_n\| \le Cn^{-r} \textit{for all} \quad n \in \mathbb{N},$$

This brings us to the main question of the talk.

WHAT IS THE CONNECTION BETWEEN THESE THEOREMS?

In other words, is there any common theme that leads to each of these theorems as a special case? The answer is yes and this connection/common theme is provided by the theory of Banach algebras. In the next section, we review some basic concepts from the theory of Banach algebras that are needed to understand this connection. More information about above theorems and related issues can be found in [3], [4] and [9]. The last section contains the details about this common theme.

2. Banach algebras

Our objects of interest are spectra of elements in a Banach algebra. We begin with the definition of a complex algebra.

Definition 2.1. Complex Algebras
A *complex algebra* A is a ring that is also a complex vector space such that

$$(\alpha a)b = \alpha(ab) = a(\alpha b) \quad \text{for all} \quad a, b \in A,\ \alpha \in \mathbb{C}$$

A is called *commutative* if $ab = ba$ for all $a, b \in A$.
We shall assume that A has a unit element 1 satisfying $1a = a = a1$ for all $a \in A$.

Definition 2.2. Banach algebras
Let A be a complex algebra. An *algebra norm on A* is a function $\|.\| : A \to \mathbb{R}$ satisfying:

(1) $\|a\| \geq 0$ for all $a \in A$ and $\|a\| = 0$ if and only if $a = 0$.
(2) $\|\alpha a\| = |\alpha|\|a\|$ for all $a \in A$ and $\alpha \in \mathbb{R}$
(3) $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in A$.
(4) $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

A *complex normed algebra* is a complex algebra A with an algebra norm defined on it. A *Banach algebra* is a complete normed algebra.

We shall assume that A is *unital*, that is A has unit 1 with $\|1\| = 1$.
Next we recall some standard examples of Banach algebras.

Example 2.3. Algebras of functions
Let X be a compact Hausdorff space, and let $C(X)$ denote the set of all complex valued continuous functions on X. Then $C(X)$ is a commutative Banach algebra under pointwise operations and the sup norm given by

$$\|f\| := \sup\{|f(x)| : x \in X\}, \quad f \in C(X)$$

Example 2.4. Algebras of Operators
Let H be a complex Hilbert space and let $BL(H)$ denote the set of all bounded(continuous) linear operators on H. Then $BL(H)$ is a Banach algebra under the usual operations and the operator norm given by

$$\|T\| := \sup\{\|T(x)\| : x \in H, \|x\| \leq 1\}, \quad T \in BL(H)$$

When H is of dimension n, $BL(H)$ can be identified with $\mathbb{C}^{n \times n}$, the algebra of all matrices of order $n \times n$ with complex entries.

More examples and basic theory of Banach algebras can be found in the following books: [1] and [7].
We now define our main objects of interest.

Definition 2.5. Spectrum
Let A be a complex Banach algebra with unit 1 and let $a \in A$. The *spectrum* $\sigma_A(a)$ of a is defined to be the set of all complex numbers λ such that $\lambda 1 - a$ is not invertible in A.
The *Spectral Radius* $r(a)$ of a is defined by

$$r_A(a) := \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$$

The subscript A will be dropped when the algebra under consideration is fixed and no confusion is likely.

Thus when $A = C(X)$ and $f \in A$, $\sigma(f)$ coincides with the range of f.

Similarly when $A = \mathbb{C}^{n\times n}$ and $M \in A$, $\sigma(M)$ is the set of all eigenvalues of A.

Properties of Spectrum:

We now recall a few well known properties of the spectrum. Let A be a complex Banach algebra with unit 1 and let $a \in A$. Then,

- $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$.
- The Spectral Radius Formula:
$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$
- The map $a \to \sigma(a)$ is upper semicontinuous. This means that given any open subset U of $\mathbb{C}$ containing $\sigma(a)$, there exists $\delta > 0$ such that for every $b \in A$ with $\|a - b\| < \delta$, $\sigma(b) \subseteq U$.

3. Inverse-closed subalgebras

We are now in a position to say something about the main question posed in the Introduction. We may note that each of the theorems mentioned there deals with some elements in some Banach algebra. In particular, each theorem says that if an element in a Banach algebra has a a particular property, then its inverse, if exists, also has the same property. This observation naturally leads to the following definition.

Definition 3.1. Inverse- closedness

Let A be a complex Banach algebra with unit 1 and let B be a subalgera of A containing 1. Then B is called *inverse-closed in* A if

$a \in B$ and $a^{-1} \in A$ implies $a^{-1} \in B$.

Now the main question posed in the Introduction can be reformulated as follows:

WHEN IS B INVERSE-CLOSED IN A?

Before discussing this question further, we may note that this concept has been given some other names also in the literature.

Let A and B be as above.

- B is inverse-closed in A.
- (B, A) is a Wiener pair.(Naimark)
- B is a spectral subalgebra of A. (Palmer)
- B is a local subalgebra of A.
- Spectral invariance, spectral permanece(Arveson)

The justification for the last of these names is due to the following characterizartion of inverse-closedness in terms of the spectrum.

Theorem 3.2. *Let A be a complex Banach algebra with unit* 1 *and let B be a subalgera of A containing* 1. *Then B is inverse-closed in A if and only if for every $x \in B$,*
$\sigma_B(x) = \sigma_A(x)$.

The following theorem gives a condition for the two spectra to coincide. More information can be found in [4].

Theorem 3.3. *Let A be a complex Banach algebra with unit* 1. *Let $Inv(A)$ denote the set of all invertible elements in A. Then, for each $a \in A$,*
$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin Inv(A)\}$ *Next, let B be a closed subalgera of A containing* 1. *Then ,*

1. *$Inv(B)$ is a union of components of $B \cap Inv(A)$.*
2. *For $x \in B$, $\sigma_B(x)$ is the union of $\sigma_A(x)$ and a (posssibly empty) collection of bounded components of the complement of $\sigma_A(x)$.*

In particular, if the the complement of $\sigma_A(x)$ is connected, then
$\sigma_B(x) = \sigma_A(x)$..

A proof of the above theorem can be found in [8]. The situation is better when the Banach algebra under consideration has some additional structure, namely involution. In particular, if it is a $B^*-algebra$, we can give a very simple condition for a subalgebra to be inverse-closed. We recall some relevant definitions.

Definition 3.4. Involutions

An *involution* on a complex algebra A is a mapping $a \to a^*$ of A into A that satisfies the following axioms:

1. $(a+b)^* = a^* + b^*$ for all a and b in A;
2. $(\alpha a)^* = \overline{\alpha} a^*$ for all a in A and α in $\mathbb{C}$;
3. $(ab)^* = b^* a^*$ for all a and b in A;
4. $(a^*)^* = a$ for all a in A.

Definition 3.5. $B^* - algebras$

A Banach algebra A with an involution $x \to x^*$ is called a $B^* - algebra$ if

$$\|x^* x\| = \|x\|^2$$

for every $x \in A$.

These algebras are also known as $C^* - algebra$ Known fact: If A is a $B^* - algebra$, then $\sigma_A(xx^*) \subseteq [0, \infty)$ for every $x \in A$. (See [8])

We now give the simple condition mentioned above.

Theorem 3.6. *(See* [8]*) Suppose A is a $B^* - algebra$ with unit* 1, *B is a closed subalgebra of A, $1 \in B$ and $x^* \in B$ for every $x \in B$. Then $\sigma_B(x) = \sigma_A(x)$ for every $x \in B$.*
In other words, B is inverse-closed in A.

Proof. : Suppose $x \in B$ has inverse in A. Then x^* and hence xx^* also have inverses in A. Hence $\sigma_A(xx^*) \subseteq (0, \infty)$. Thus the complement of $\sigma_A(xx^*)$ is connected. This implies $\sigma_B(xx^*) = \sigma_A(xx^*) \subseteq (0, \infty)$. Hence $(xx^*)^{-1} \in B$ and consequently $x^{-1} = x^*(xx^*)^{-1} \in B$. □

Remark 3.7. We are now in a position to explain the connection between the three theorems stated in the introduction. In fact, as explained below each of these theorems is a special case of Thorem 3.6.

(1) Let $A = C(\Gamma)$, where Γ denotes the unit circle and B be the set of continuous functions in A with absolutely convergent Fourier series. We have already seen in Example 2.3 that A is a Banach algebra. It is routine to check that B is a closed subalgebra of A. The map $f \to \overline{f}$, where $\overline{f}$ denotes the complex conjugate of f, is an involution on A making it a $B^* - algebra$. Also, it is easy to prove that if the Fourier series of f converges absolutely, then so does that of $\overline{f}$. In other words, B satisfies the hypotheses of Theorem 3.6 and is hence an inverse-closed subalgebra of A. This is Wiener's theorem (Thorem 1.1).

(2) Let $A = BL(\ell^2(\mathbb{Z}))$ and B be the set of all matrices satisfying the off diagonal decay conditions given in Jaffard's theorem (Thorem1.2) . Then A is a Banach algebra, as seen in Example 2.4. It requires some work to check that B is a closed subalgebra of A. Next, for $T \in A$, let T^* denote the adjoint of T. Then the map $T \to T^*$ is an involution on A and A is $B^* - algebra$ with respect to this involution. This is well known and can be found in many books, for example, [7], [8]. Further, it is also well known that if $\alpha_{i,j}$ is the $(i,j)th$ entry of the matrix of T, then the $(i,j)th$ entry of the matrix of T^* is $\overline{\alpha_{j,i}}$. Hence if T satisfies the off diagonal decay conditions, then so does T^*. Thus B satisfies the hypotheses of Theorem 3.6 and is hence an inverse-closed subalgebra of A. This is precisely the statement of Jaffard's theorem (Theorem 1.2).

(3) Let $A = BL(\ell^2(\mathbb{Z}))$ and B be the set of all band dominated matrices. Then, as above, A is $B^* - algebra$ and B is a closed subalgebra of A. Also, if $T \in A$ is band dominated, then so is T^*. Thus again by Therem 3.6, it follows that B is an inverse-closed subalgebra of A. This implies Theorem 1.4.

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