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LINEAR MAPS PRESERVING PSEUDOSPECTRUM AND CONDITION SPECTRUM

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ABSTRACT. We discuss properties of pseudospectrum and condition spectrum of an element in a complex unital Banach algebra and its ϵ-perturbation. Several results are proved about linear maps preserving pseudospectrum/ condition spectrum. These include the following:

(1) Let A, B be complex unital Banach algebras and $\epsilon > 0$. Let $\Phi : A \to B$ be an ϵ-pseudospectrum preserving linear onto map. Then Φ preserves spectrum. If A and B are uniform algebras, then, Φ is an isometric isomorphism.
(2) Let A, B be uniform algebras and $0 < \epsilon < 1$. Let $\Phi : A \to B$ be an ϵ-condition spectrum preserving linear map. Then Φ is an $\epsilon^{'}$-almost multiplicative map, where $\epsilon, \epsilon^{'}$ tend to zero simultaneously.

1. INTRODUCTION

Let A be a complex Banach algebra with unit 1. We shall identify $\lambda.1$ with λ. We recall that the spectrum of an element $a \in A$ is defined as

$$\sigma(a) = \left\{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\right\},$$

where A^{-1} is the set of all invertible elements of A. The spectral radius of an element a is defined as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

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There are several extensions and generalizations of the concept of spectrum. These include the Ransford's generalized spectrum [18], ϵ-pseudospectrum [21] and the ϵ-condition spectrum [13]. Unlike the spectrum, which is a purely algebraic concept, both the ϵ-pseudospectrum and ϵ-condition spectrum depend on the norm. Also both these sets contain the spectrum as a subset.

Linear preserver problems (LPP) is an active research area in matrix and operator theory. A brief discussion on LPP can be found in [3]. The monograph by Molnar [15] contains a wealth of information about such problems. The most popular among these is the problem of characterizing spectrum preserving maps, studied by many authors [14, 7, 20]. Let X, Y be Banach spaces, $BL(X)$ denote the algebra of all bounded linear operators on X and X^* denote the dual of X. Jafarian and Sourour proved [7] that a spectrum preserving linear map Φ from $BL(X)$ onto $BL(Y)$ is either of the form $\Phi(T) = ATA^{-1}, T \in BL(X)$ for an isomorphism A from X onto Y or of the form $\Phi(T) = BT^*B^{-1}, T \in BL(X)$ for an isomorphism B of X^* onto Y. Thus Φ is multiplicative or anti-multiplicative.

In this paper we study the linear maps which preserve ϵ-pseudospectrum and ϵ-condition spectrum between complex unital Banach algebras. One of the surprises here is that it turns out that a map preserving ϵ-pseudospectrum also preserves spectrum (Theorem 3.10). In many cases such a map simply becomes an isometric isomorphism (Remark 3.12).
In section 2, we give definitions of ϵ-pseudospectrum, ϵ-condition spectrum, ϵ-almost multiplicative map and ϵ-isometry. We establish relations between pseudospectrum and condition spectrum of an element in a complex unital Banach algebra (Proposition 2.5, 2.6). In Section 3, we prove that any linear map between complex unital Banach algebras which preserves ϵ-pseudospectrum also preserves spectrum (Theorem 3.10). We also prove an analogue of the Gleason–Kahane–Zelazko theorem for ϵ-pseudospectrum (Theorem 3.13). In section 4, we discuss ϵ-condition spectrum preserving maps in complex unital Banach algebras. We establish that ϵ-condition spectrum is closely related to ϵ-almost multiplicative map (Theorem 4.4). In section 5, we study ϵ-perturbation of a complex unital Banach algebra. We prove various properties and relations between spectrum, pseudospectrum and condition spectrum of an element in a Banach algebra and its ϵ-perturbation.

2. Preliminaries

In this section we introduce some definitions and terminology used in this paper. A relation connecting pseudospectrum and condition spectrum of an element in a complex unital Banach algebra is given. We also show that a small perturbation of an isomorphism is an almost multiplicative map.

Definition 2.1. (ϵ-pseudospectrum) Let A be a complex unital Banach algebra with unit 1 and $\epsilon > 0$. The *ϵ-pseudospectrum* of an element $a \in A$ is denoted by $\Lambda_\epsilon(a)$ and is defined as,

$$\Lambda_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \ \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon} \right\},$$

with the convention that $\|(\lambda - a)^{-1}\| = \infty$ if $\lambda - a$ is not invertible.

Note that because of this convention, $\sigma(a) \subseteq \Lambda_\epsilon(a)$ for every $\epsilon > 0$. For more information on ϵ-pseudospectrum one may refer to [21].

Definition 2.2. (ϵ-condition spectrum) Let A be a complex unital Banach algebra with unit 1 and $0 < \epsilon < 1$. The *ϵ-condition spectrum* of an element $a \in A$ is denoted by $\sigma_\epsilon(a)$ and is defined as,

$$\sigma_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \ \|\lambda - a\| \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon} \right\},$$

with the convention that $\|\lambda - a\| \|(\lambda - a)^{-1}\| = \infty$, if $\lambda - a$ is not invertible.

Here also $\sigma(a) \subseteq \sigma_\epsilon(a)$ for $0 < \epsilon < 1$. One may refer to [13] for examples and elementary properties of the ϵ-condition spectrum.

Definition 2.3. Let A, ϵ be as in Definition 2.2. The ϵ-condition spectral radius $r_\epsilon(a)$ of an element $a \in A$ is defined as

$$r_\epsilon(a) := \sup\{|\lambda| : \lambda \in \sigma_\epsilon(a)\}.$$

Remark 2.4. Let A, ϵ be as in Definition 2.2 then,

$$r(a) \leq r_\epsilon(a) \leq \frac{1+\epsilon}{1-\epsilon} \|a\|.$$

(see Theorem 2.9 of [13]).

Next two propositions establish a relationship between condition spectrum and pseudospectrum of an element in a complex unital Banach algebra. These propositions provide an answer to a question raised by the authors in [1] (see Remark 4.4 of [1]).

Proposition 2.5. *Let A be a complex Banach algebra with unit 1, $a \in A$ and $0 < \epsilon < 1$. Then*

$$\sigma_\epsilon(a) \subseteq \Lambda_{\frac{2\epsilon\|a\|}{1-\epsilon}}(a).$$

Proof. Let $\lambda \in \sigma_\epsilon(a)$. Then $|\lambda| \leq \dfrac{(1+\epsilon)\|a\|}{1-\epsilon}$ and hence

$$\|\lambda - a\| \ \ \leq \ \ |\lambda| + \|a\| \leq \frac{(1+\epsilon)\|a\|}{1-\epsilon} + \|a\| = \frac{2\|a\|}{1-\epsilon}.$$

Since $\lambda \in \sigma_\epsilon(a)$, we have

$$\|\lambda - a\| \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}.$$

Thus

$$\begin{aligned} \|(\lambda - a)^{-1}\| &\geq \frac{1}{\epsilon\|\lambda - a\|} \\ &\geq \frac{1-\epsilon}{2\epsilon\|a\|}. \end{aligned}$$

$\square$

Proposition 2.6. *Let A be a complex Banach algebra with unit 1 and $\epsilon > 0$. Suppose $a \in A$ is not a scalar multiple of 1 and let $M_a := \inf\{\|z - a\| : \quad z \in \mathbb{C}\}$. Then $\Lambda_\epsilon(a) \subseteq \sigma_{\frac{\epsilon}{M_a}}(a)$.*

Proof. Suppose $\lambda \in \Lambda_\epsilon(a)$. Then,

$$\|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}.$$

Also,

$$\|\lambda - a\| \geq \inf\{\|z.1 - a\| : z \in \mathbb{C}\} = M_a > 0.$$

Hence

$$\|\lambda - a\|\|(\lambda - a)^{-1}\| \geq \frac{M_a}{\epsilon}.$$

□

Remark 2.7. If $a = \mu.1$ for some $\mu \in \mathbb{C}$, then ϵ-condition spectrum of a is the singleton set $\{\mu\}$ and ϵ-pseudospectrum is the closed ball with center μ and radius ϵ. Thus the condition on a can not be dropped from the above proposition.

Definition 2.8. Let A, B be complex Banach algebras and $\epsilon > 0$. A linear map $\Phi : A \to B$ is said to be

(1) an *ϵ-almost multiplicative map*, if
$$\|\Phi(ab) - \Phi(a)\Phi(b)\| \leq \epsilon\|a\|\|b\| \text{ for all } a, b \in A.$$
(2) an *ϵ-almost anti-multiplicative map*, if
$$\|\Phi(ab) - \Phi(b)\Phi(a)\| \leq \epsilon\|a\|\|b\| \text{ for all } a, b \in A.$$
(3) an *ϵ-almost Jordan multiplicative map*, if
$$\|\Phi(a^2) - \Phi(a)^2\| \leq \epsilon\|a\|^2 \text{ for all } a \in A.$$

See [8] for more information on such maps. In [8] ϵ-almost multiplicative maps are called ϵ-isomorphisms. We have avoided this terminology because the word isomorphism usually includes the assumption of injectivity. We have not made such an assumption here. If $B = \mathbb{C}$, Φ is called ϵ-almost multiplicative linear functional. It is obvious that every ϵ-almost multiplicative or ϵ- almost anti-multiplicative map is ϵ- almost Jordan multiplicative map. See [1] for a detailed discussion of the converse of this, called approximate Herstein's theorem.

Definition 2.9. (ϵ-isometry) Let A, B be complex Banach spaces and $\epsilon > 0$. A linear continuous injective map $\Phi : A \to B$ is said to be an *ϵ-isometry* , if

$$\|\Phi\| \leq 1 + \epsilon \quad \text{and} \quad \|\Phi^{-1}\| \leq 1 + \epsilon.$$

One may refer to [8] for more information on ϵ- isometry.

Example 2.10. (ϵ-almost multiplicative map) Consider $\mathbb{C}^{2\times 2}$ as the set of all bounded linear maps on $(\mathbb{C}^2, \|\cdot\|_2)$. For $0 < \epsilon < \dfrac{1}{4}$ consider $\Phi : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ defined by

$$\Phi(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = \begin{bmatrix} a_{11} & a_{12}(1+\epsilon) \\ a_{21}(1+\epsilon) & a_{22} \end{bmatrix}.$$

We can write the above as $\Phi(A) = A + \Psi(A)$, where $\Psi : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ is a linear map given by

$$\Psi(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = \begin{bmatrix} 0 & a_{12}\epsilon \\ a_{21}\epsilon & 0 \end{bmatrix}.$$

So that $\|\Psi\| \leq \epsilon$. Now for $A, B \in \mathbb{C}^{2\times 2}$ we have

$$\begin{aligned} \|\Phi(A.B) - \Phi(A).\Phi(B)\| &= \|\Psi(A.B) - \Psi(A).B - A.\Psi(B) - \Psi(A).\Psi(B)\| \\ &\leq 4\|\Psi\|\|A\|\|B\| \\ &\leq 4\epsilon\|A\|\|B\|. \end{aligned}$$

Thus Φ is a 4ϵ-almost multiplicative map.

The following proposition shows that a small perturbation of a homomorphism is an almost multiplicative map.

Proposition 2.11. *Let A, B be complex Banach algebras, $\Phi : A \to B$ be a continuous homomorphism, $0 < \epsilon < 1$ and $\Psi : A \to B$ a bounded linear map with $\|\Psi\| \leq min\left\{\frac{\epsilon}{4}, \frac{\epsilon}{4\|\Phi\|}\right\}$. Then $\Phi + \Psi$ is an ϵ-almost multiplicative map.*

Proof. For $a, b \in A$ we have,

$$\begin{aligned} &\|(\Phi + \Psi)(ab) - (\Phi + \Psi)(a)(\Phi + \Psi)(b)\| \\ &\quad= \|\Phi(ab) + \Psi(ab) - (\Phi(a) + \Psi(a))(\Phi(b) + \Psi(b))\| \\ &\quad= \|\Phi(ab) - \Phi(a)\Phi(b) + \Psi(ab) - \Psi(a)\Psi(b) - \Phi(a)\Psi(b) - \Psi(a)\Phi(b)\| \\ &\quad\leq (\|\Psi\| + \|\Psi\|^2 + 2\|\Phi\|\|\Psi\|)\|a\|\|b\| \\ &\quad\leq \epsilon\|a\|\|b\|. \end{aligned}$$

∎

Remark 2.12. A special case of this when $B = \mathbb{C}$ was proved by Johnson [10]. It is also easy to prove in a similar way that if Φ is an anti-homomorphism then $\Phi + \Psi$ is ϵ-almost anti-multiplicative and if Φ is a Jordan multiplicative map then $\Phi + \Psi$ is ϵ-almost Jordan multiplicative. This raises a natural question: Is every ϵ-almost multiplicative map close to a multiplicative map? This is true for certain pairs of Banach algebras known as AMNM pairs introduced by Johnson [11]. See also [1] for a discussion of similar questions in case of anti-homomorphisms.

3. Linear maps preserving pseudospectrum

We begin this section by giving sufficient conditions for a map between complex unital Banach algebras to preserve pseudospectrum and condition spectrum. Then we prove that every pseudospectrum preserving map between complex unital Banach algebras preserves spectrum also. This has some interesting consequences. This section also contains an analogue of the Gleason–Zelazko theorem for pseudospectrum.

Theorem 3.1. *Let A, B be complex Banach algebras with units $1_A, 1_B$ respectively. Suppose $\Phi : A \to B$ is*

(1) *bijective;*

(2) *linear;*
(3) *unital (i.e* $\Phi(1_A) = 1_B$*);*
(4) *either multiplicative (that is,* $\Phi(ab) = \Phi(a)\Phi(b)$ *for all* $a, b \in A$*) or anti-multiplicative (that is,* $\Phi(ab) = \Phi(b)\Phi(a)$ *for all* $a, b \in A$*);*
(5) *isometry.*

Then

$$\Lambda_\epsilon(a) = \Lambda_\epsilon(\Phi(a)) \quad for \ \ all \ \ a \in A, \ \ \epsilon > 0, and$$
$$\sigma_\epsilon(a) = \sigma_\epsilon(\Phi(a)) \quad for \ \ all \ \ a \in A, \ \ 0 < \epsilon < 1.$$

Proof. Let $a \in A$ and $\epsilon > 0$. If $\lambda \in \Lambda_\epsilon(a)$, then

$$\|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}.$$

Since Φ is an isometry,

$$\|\Phi[(\lambda - a)^{-1}]\| \geq \frac{1}{\epsilon}.$$

Since Φ is multiplicative(or anti-multiplicative) and unital, inverses are preserved. Hence

$$\|[\Phi(\lambda - a)]^{-1}\| \geq \frac{1}{\epsilon}.$$

Since Φ is unital and linear,

$$\|[\lambda - \Phi(a)]^{-1}\| \geq \frac{1}{\epsilon}.$$

Hence $\lambda \in \Lambda_\epsilon(\Phi(a))$ and $\Lambda_\epsilon(a) \subseteq \Lambda_\epsilon(\Phi(a))$. By symmetry we can show that $\Lambda_\epsilon(\Phi(a)) \subseteq \Lambda_\epsilon(a)$. The same argument also shows that $\sigma_\epsilon(a) = \sigma_\epsilon(\Phi(a))$ for all $a \in A$, because $\|\lambda - a\| = \|\lambda - \Phi(a)\|$. □

Remark 3.2. If Φ is assumed to be continuous(instead of isometry) in Theorem 3.1, then we can prove $\Lambda_{\frac{\epsilon}{\|\Phi\|}}(\Phi(a)) \subseteq \Lambda_\epsilon(a)$ and $\sigma_{\frac{\epsilon}{\|\Phi\|^2}}(\Phi(a)) \subseteq \sigma_\epsilon(a)$.

Theorem 3.3. *Let T be a Laurent operator on $l^2(\mathbb{Z})$ with a continuous symbol f. Then for each $\epsilon > 0$ $\Lambda_\epsilon(T) = \Lambda_\epsilon(f)$ and for $0 < \epsilon < 1$ $\sigma_\epsilon(T) = \sigma_\epsilon(f)$, where f is regarded as an element of $C(\Gamma)$, where Γ is the unit circle.*

Proof. Let A be the Banach algebra of all Laurent operators on $l^2(\mathbb{Z})$ (see [4, 5]) with continuous symbols and $B = C(\Gamma)$. First note that if $T \in A$ and T is invertible in $l^2(\mathbb{Z})$, then $T^{-1} \in A$ [5]. Hence the ϵ-pseudospectrum of T regarded as an element of A is the same as the ϵ-pseudospectrum of T regarded as an element of $BL(l^2(\mathbb{Z}))$. The same can be said about ϵ-condition spectrum. The map $\Phi : A \to B$ defined by,

$$\Phi(T) = \text{symbol}(T),$$

is linear, bijective, unital, multiplicative and isometry([5], Theorem 1.27, 1.28). Hence the result follows from Theorem 3.1. □

Remark 3.4. The above theorem gives a very useful technique to compute the ϵ-pseudospectra and ϵ-condition spectra of Laurent operators with continuous symbols. See Example 2.13 in [13] where the ϵ-condition spectrum of the bilateral shift V on $l^2(\mathbb{Z})$ is computed as

$$\sigma_\epsilon(V) = \left\{\lambda \in \mathbb{C} : \frac{1-\epsilon}{1+\epsilon} \leq |\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\right\}.$$

Using the calculations given there we can show that the ϵ-pseudospectrum is given by

$$\Lambda_\epsilon(V) = \{\lambda \in \mathbb{C} : 1-\epsilon \leq |\lambda| \leq 1+\epsilon\}.$$

Remark 3.5. If A, B are uniform algebras, then condition(4) can be dropped from Theorem 3.1 in view of Nagasawa's theorem [16].

Definition 3.6. (standard operator algebra) A standard operator algebra R on a Banach space X is a Banach subalgebra of $BL(X)$ which contains the identity and the ideal of all finite rank operators.

Corollary 3.7. *Let X, Y be Banach spaces and A, B be standard operator algebras on X, Y respectively. Let $\Phi : A \to B$ be a linear, bijective, spectrum preserving isometry, then Φ preserves ϵ-pseudospectrum for every $\epsilon > 0$. Also Φ preserves ϵ-condition spectrum for every $0 < \epsilon < 1$.*

Proof. The only hypothesis of Theorem 3.1 not included here is (4). This follows from a result of Sourour [20]. $\square$

Remark 3.8. In [2], the authors introduce thirteen parts of the spectrum of an operator and show that if Φ preserves any one of the thirteen parts of the spectrum, then Φ is multiplicative or anti-multiplicative. In view of this Corollary 3.7 can be strengthened by assuming that Φ preserves one of the thirteen parts.

Remark 3.9. A similar result for the case When $A = B = \mathbb{C}^{n\times n}$ was proved in Theorem 3.2 [9].

Theorem 3.10. *Let A, B be complex Banach algebras and $\epsilon > 0$. Let $\Phi : A \to B$ be an ϵ-pseudospectrum preserving linear onto map. Then Φ preserves spectra of elements.*

Proof. We have

$$\Lambda_\epsilon(a) = \Lambda_\epsilon(\Phi(a)) \quad for \quad all \;\; a \in A.$$

Suppose $\lambda \notin \sigma(a)$, choose $t > \epsilon\|(\lambda - a)^{-1}\|$. Then

$$\|[t(\lambda - a)]^{-1}\| < \frac{1}{\epsilon}.$$

Thus

$$t\lambda \notin \Lambda_\epsilon(ta) = \Lambda_\epsilon(\Phi(ta)) \supseteq \sigma(\Phi(ta)) = t\sigma(\Phi(a))$$

so

$$\lambda \notin \sigma(\Phi(a)).$$

Therefore

$$\sigma(\Phi(a)) \subseteq \sigma(a).$$

In a similar way we can prove that

$$\sigma(a) \subseteq \sigma(\Phi(a)).$$

Hence

$$\sigma(\Phi(a)) = \sigma(a).$$

□

Corollary 3.11. *Let $BL(X), BL(Y)$ be the algebra of all bounded linear operators on the Banach spaces X, Y respectively. Let $\Phi : BL(X) \to BL(Y)$ be an ϵ- pseudospectrum preserving linear onto map. Then $\Phi(T) = ATA^{-1}$ for an isomorphism A of X onto Y or $\Phi(T) = BT^*B^{-1}$ for an isomorphism B of X^* onto Y, where X^* denote the dual of X. In particular Φ is multiplicative or anti-multiplicative.*

Proof. Follows by Theorem 3.10 and a result of Jafarian and Sourour [7]. □

Remark 3.12. In Theorem 3.10, if A and B are uniform algebras, then T is an isometry, because for each $a \in A$, $\|T(a)\| = r(T(a)) = r(a) = \|a\|$. But then T becomes an isomorphism in view of Nagasawa's theorem [8, 16]. Thus T is an isometric isomorphism.

The following theorem is analogous to the classical Gleason–Zelazko theorem [22], [17].

Theorem 3.13. *Let A be a complex commutative Banach algebra with unit 1 and $\epsilon > 0$. Let $\phi : A \to \mathbb{C}$ be a linear functional such that $\phi(1) = 1$ and $\phi(a) \in \Lambda_\epsilon(a)$ for all $a \in A$. Then ϕ is multiplicative.*

Proof. We claim that $\phi(a) \in \sigma(a)$ for all $a \in A$. Let $\phi(a) = \lambda$. If $\lambda \notin \sigma(a)$ then $\lambda - a \in A^{-1}$. Choose $t > \epsilon\|(\lambda - a)^{-1}\|$, then $\|(\lambda t - ta)^{-1}\| < \frac{1}{\epsilon}$. Thus $t\lambda = \phi(ta) \notin \Lambda_\epsilon(ta)$. This gives a contradiction. Now the conclusion follows from the Gleason–Zelazko theorem. □

Remark 3.14. In Theorem 3.13, if we replace the hypothesis "$\phi(a) \in \Lambda_\epsilon(a)''$ by "$\phi(a) \in \sigma_\epsilon(a)''$, then ϕ is only almost multiplicative. This is proved in [12].

Theorem 3.15. *Let A, B be complex unital Banach algebras and $\Phi : A \to B$ be an ϵ-pseudospectrum preserving linear map for some $\epsilon > 0$. Suppose Φ is multiplicative or anti-multiplicative. Then Φ preserves norms of all invertible elements of A.*

Proof. Suppose there exist $a \in A^{-1}$ such that $\|\Phi(a^{-1})\| \neq \|a^{-1}\|$. Assume $\|a^{-1}\| < \|\Phi(a^{-1})\|$, choose $t > 0$ such that

$$\epsilon\|a^{-1}\| < t \leq \epsilon\|\Phi(a^{-1})\| = \epsilon\|[\Phi(a)]^{-1}\|.$$

then

$$\|(ta)^{-1}\| < \frac{1}{\epsilon}.$$

Thus $0 \notin \Lambda_\epsilon(ta)$. But

$$\|[\Phi(ta)]^{-1}\| \geq \frac{1}{\epsilon}$$

implies $0 \in \Lambda_\epsilon(\Phi(ta))$. This contradicts the fact that Φ preserves ϵ-pseudospectrum. □

Corollary 3.16. *Let $A = M_n(\mathbb{C})$ the algebra of all $n\times n$ matrices. Let $\Phi : A \to A$ be a linear, bijective map. Then the following are equivalent.*

(1) *Φ preserves ϵ-pseudospectrum for some $\epsilon > 0$.*
(2) *Φ is an isometric isomorphism or anti-isomorphism.*
(3) *Φ preserves δ-pseudospectrum for every $\delta > 0$.*

Proof. (1) $\Rightarrow$ (2), By Theorem 3.10, if Φ preserves ϵ-pseudospectrum it preserves spectrum. Hence by Corollary 3.11, Φ is an isomorphism or anti-isomorphism. Since the set of all invertible elements is dense in $M_n(\mathbb{C})$ [6], Theorem 3.15 implies that Φ is an isometry.
(2) $\Rightarrow$ (3) by Theorem 3.1.
(3) $\Rightarrow$ (1), trivial. □

Corollary 3.17. *Let A, B be complex uniform algebras. Let $\Phi : A \to B$ be a linear bijective map. Then the following are equivalent.*

(1) *Φ preserves spectrum*
(2) *Φ is an isometric isomorphism.*
(3) *Φ preserves ϵ-pseudospectrum for every $\epsilon > 0$.*
(4) *Φ preserves ϵ-pseudospectrum for some $\epsilon > 0$.*

Proof. (1) $\Rightarrow$ (2), Suppose Φ preserves spectrum. Since A is a uniform algebra, we have $r(a) = \|a\|$ for all $a \in A$. So Φ is an isometry and hence by Nagasawa Theorem Φ is an isomorphism.
(2) $\Rightarrow$ (3) by Theorem 3.1.
(3) $\Rightarrow$ (4), trivial.
(4) $\Rightarrow$ (1), follows from Theorem 3.10. □

Remark 3.18. In Remark 4.4 of [1] Alaminos *et.al.* were unable to get how Theorem 4.2 of [1] is related to Theorem 5 of [12]. We can now comment on this relationship. Recall that in Theorem 5 of [12] it is assumed that A is a complex commutative Banach algebra, $0 < \epsilon < \frac{1}{3}$ and $\Phi(a) \in \sigma_\epsilon(a)$ for all $a \in A$. In view of Proposition 2.5, this implies $\Phi(a) \in \sigma_\epsilon(a) \subseteq \Lambda_{\frac{2\epsilon}{1-\epsilon}}(a)$ for $a \in A$ with $\|a\| = 1$. This is the hypothesis of Theorem 4.2 of [1] and the conclusion is that Φ is δ-almost multiplicative for a suitable $\delta > 0$. Thus Theorem 5 of [12] follows from Theorem 4.2 of [1].

4. Almost multiplicative linear maps and condition spectrum

In [8] Jarosz proved that every almost multiplicative linear functional is continuous. A more general result is proved in [1] (see Proposition 3.1 of [1]). Here we prove the continuity of an almost multiplicative linear map using the condition spectrum. We discuss maps which preserve condition spectrum between complex unital Banach algebras. We show that there is a close relation between the condition spectrum and almost multiplicative maps.

Definition 4.1. (Spectrally normed algebra) Let A be a complex Banach algebra with unit 1. A is said to be a *spectrally normed algebra* if there exist a $k \geq 1$ such that $\|a\| \leq k \; r(a) \;$ for all $\; a \in A$. Here $r(a)$ denotes the spectral radius of a.

See [17] for more information on spectrally normed algebras.

Theorem 4.2. *Let A, B be unital Banach algebras and $0 < \epsilon < 1$. Let $\Phi : A \to B$ be a unital (i.e $\Phi(1) = 1$) ϵ-almost multiplicative or ϵ-almost anti-multiplicative linear map. Then $\sigma(\Phi(a)) \subseteq \sigma_\epsilon(a)$ for all $a \in A$. In particular, $r(\Phi(a)) \leq r_\epsilon(a) \leq \dfrac{1+\epsilon}{1-\epsilon}\|a\|$. If, in addition, B is also spectrally normed, then Φ is continuous and $\|\Phi\| \leq k(1+\epsilon)/(1-\epsilon)$ for some constant $k > 0$.*

Proof. Let $\lambda \notin \sigma_\epsilon(a)$ then $\lambda - a$ is invertible and

$$\|(\lambda - a)\|\|(\lambda - a)^{-1}\| < \frac{1}{\epsilon}.$$

Thus,

$$\begin{aligned}
\|1 - \Phi(\lambda - a)\Phi((\lambda - a)^{-1})\| &= \|\Phi(1) - \Phi(\lambda - a)\Phi(\lambda - a)^{-1}\| \\
&= \|\Phi((\lambda - a)(\lambda - a)^{-1}) - \Phi(\lambda - a)\Phi(\lambda - a)^{-1}\| \\
&\leq \epsilon\|\lambda - a\|\|(\lambda - a)^{-1}\| \\
&< 1.
\end{aligned}$$

Hence $\Phi(\lambda - a)\Phi((\lambda - a)^{-1})$ is invertible and therefore, $\Phi(\lambda - a) = \lambda - \Phi(a)$ is invertible, i.e $\lambda \notin \sigma(\Phi(a))$.
If B is spectrally normed, then there exist $k > 0$ such that

$$\|b\| \leq k \; r(b) \quad \text{for all} \quad b \in B\,.$$

Hence for all $a \in A$,

$$\|\Phi(a)\| \;\; \leq \;\; k \;\; r(\Phi(a)) \leq k\frac{1+\epsilon}{1-\epsilon}\|a\|.$$

□

Remark 4.3. A special case of this result, when $B = \mathbb{C}$, was proven in ([12], Theorem 3).

Theorem 4.4. *Let A, B be complex Banach algebras and $0 < \epsilon < 1$. Let $\Phi : A \to B$ be an ϵ-condition spectrum preserving linear map. Then,*
(i) Φ is injective and unital.
(ii) If B is a uniform algebra, then Φ is continuous and $\|\Phi\| \leq \dfrac{1+\epsilon}{1-\epsilon}$.
(iii) If A, B are both uniform algebras and Φ is onto then Φ is invertible. In fact it is $\dfrac{2\epsilon}{1-\epsilon}$-isometry and also ϵ'-almost multiplicative map, where ϵ, ϵ' tend to zero simultaneously.

Proof. (i) $\Phi(a) = 0 \implies \sigma_\epsilon(a) = \sigma_\epsilon(\Phi(a)) = \sigma_\epsilon(0) = \{0\} \implies a = 0$(see [13], Corollary 3.2). Also

$$\sigma_\epsilon(\Phi(1)) = \sigma_\epsilon(1) = \{1\} \implies \Phi(1) = 1.$$

(ii) Since Φ preserves ϵ-condition spectrum,

$$\sigma(\Phi(a)) \subseteq \sigma_\epsilon(\Phi(a)) = \sigma_\epsilon(a) \quad \text{for all} \quad a \in A.$$

Since B is a function algebra, we get

$$\|\Phi(a)\| = r(\Phi(a)) \leq \frac{1+\epsilon}{1-\epsilon}\|a\| \quad \text{for all} \quad a \in A \quad \text{(by Theorem 4.2)}.$$

(iii) From (ii) we have,

$$\|\Phi\| \leq \frac{1+\epsilon}{1-\epsilon} = 1 + \frac{2\epsilon}{1-\epsilon}.$$

In a similar way we can prove that

$$\|\Phi^{-1}\| \leq \frac{1+\epsilon}{1-\epsilon} = 1 + \frac{2\epsilon}{1-\epsilon}.$$

Hence by a result due to Jarosz [8]; Φ is an ϵ'-almost multiplicative map such that ϵ, ϵ' tends to zero simultaneously. □

Theorem 4.5. *Let X and Y be super reflexive Banach spaces and $\Phi : BL(X) \to BL(Y)$ be a bijective linear map. Then for every ϵ with $0 < \epsilon < 1$ there exist $\delta > 0$ such that if $\sigma(\Phi(T)) \subseteq \sigma_\delta(T)$ for all $T \in BL(X)$, then Φ is ϵ-almost multiplicative or ϵ-almost anti-multiplicative. In particular, if $X = Y = H$, where H is a separable Hilbert space, then $\|\Phi - \Psi\| < \epsilon$ for some automorphism or anti-automorphism $\Psi : BL(H) \to BL(H)$.*

Proof. By Proposition 2.5 $\sigma(\Phi(T)) \subseteq \Lambda_{\frac{2\delta}{1-\delta}}(T)$ for $T \in BL(X)$ with $\|T\| = 1$. Now the conclusions follows from Theorem 5.8 and 5.10 of [1]. □

5. Algebraic perturbation and condition spectrum

In this section we consider an algebraic ϵ-perturbation on a Banach algebra. We prove various results connecting spectrum, pseudospectrum and condition spectrum of an element in a complex Banach algebra and its ϵ-perturbation.

Definition 5.1. Let $(A, \cdot, \|\cdot\|)$ be a complex Banach algebra with unit 1 and $\epsilon > 0$. By an algebraic ϵ-perturbation of A we mean any multiplication $*$ on the vector space A such that $(A, +, *)$ is a complex algebra and,

$$\|a * b - a \cdot b\| \leq \epsilon\|a\|\|b\| \quad \textit{for all} \quad a, b \in A.$$

See [8] for more information on ϵ-perturbation.

Theorem 5.2. *Let $(A, \cdot, \|\cdot\|)$ be a complex Banach algebra and $*$ be an algebraic ϵ-perturbation. Assume both multiplications have the same unit 1. Then the following hold:*

1. *The multiplication $* : A \times A \to A$ is continuous. It induces a norm $|||\cdot|||$ on A so that $(A, *, |||\cdot|||)$ is a Banach algebra.*
2. *$\|a\| \leq |||a||| \leq (1+\epsilon)\|a\|$ for all $a \in A$.*

(3) *The multiplication $\cdot$ is an algebraic $\epsilon(1+\epsilon)$-perturbation of $(A, *, |||\cdot|||)$.*

Proof. We refer to [19] for a proof of (1) and (2). (3) can be proved as follows. For all $a, b \in A$,

$$\begin{aligned} |||a * b - a \cdot b||| &\leq (1+\epsilon)\|a * b - a \cdot b\| \\ &\leq \epsilon(1+\epsilon)\|a\|\|b\| \\ &\leq \epsilon(1+\epsilon)|||a|||\ |||b|||\,. \end{aligned}$$

Thus the multiplication $\cdot$ is an algebraic $\epsilon(1+\epsilon)$-perturbation of $(A, *, |||\cdot|||)$. □

Corollary 5.3. *Let A be a complex Banach algebra with unit 1 and $\epsilon > 0$. Let $a \in A$ and $\|a\| < \dfrac{1}{1+\epsilon}$. Then $1-a$ is invertible in $(A, *, |||\cdot|||)$ and $\|(1-a)_*^{-1}\| \leq \dfrac{1}{1-(1+\epsilon)\|a\|}$, where a_*^{-1} denote the inverse of a in the complex Banach algebra $(A, *, |||\cdot|||)$. More generally let $a \in A$ and $\|a\| < \dfrac{|\lambda|}{1+\epsilon}$, then $\lambda - a$ is invertible in $(A, *, |||\cdot|||)$ and $\|(\lambda-a)_*^{-1}\| \leq \dfrac{1}{|\lambda| - (1+\epsilon)\|a\|}$.*

Proof. If $a \in A$ and $\|a\| < \dfrac{1}{1+\epsilon}$ then, $|||a||| \leq (1+\epsilon)\|a\| < 1$. Hence $1-a$ is invertible in $(A, *, |||\cdot|||)$. Also

$$\begin{aligned} \|(1-a)_*^{-1}\| &\leq |||(1-a)_*^{-1}||| \\ &\leq \frac{1}{1-|||a|||} \\ &\leq \frac{1}{1-(1+\epsilon)\|a\|}\,. \end{aligned}$$

□

Definition 5.4. Let A be a complex Banach algebra with unit 1 and $*$ be an algebraic ϵ-perturbation with the same unit. For $\epsilon > 0$, the ϵ-*pseudospectrum* of an element a in the new Banach algebra $(A, *, |||\cdot|||)$ is defined as

$$\Lambda_\epsilon^*(a) = \left\{\lambda \in \mathbb{C} : |||(\lambda - a)_*^{-1}||| \geq \frac{1}{\epsilon}\right\}.$$

Definition 5.5. Let A be a complex Banach algebra with unit 1 and $*$ be an algebraic ϵ-perturbation with same unit. For $0 < \epsilon < 1$, the ϵ-*condition spectrum* of an element a in the new Banach algebra $(A, *, |||\cdot|||)$ is defined as

$$\sigma_\epsilon^*(a) = \left\{\lambda \in \mathbb{C} : |||\lambda - a|||\ |||(\lambda - a)_*^{-1}||| \geq \frac{1}{\epsilon}\right\}.$$

Now we prove several relations connecting spectrum and condition spectrum between the algebras $(A, \cdot, \|\cdot\|)$ and $(A, *, |||\cdot|||)$.

Proposition 5.6. *Let A be a complex Banach algebra with unit 1 and $0 < \epsilon < 1$. Let $*$ be an ϵ-perturbation with the same unit, then $\sigma(a) \subseteq \sigma_\epsilon^*(a)$*

Proof. Note

$$\begin{aligned} \|a * b - a.b\| &\leq \epsilon \|a\| \|b\| \\ &\leq \epsilon \ |||a||| \ |||b||| \,. \end{aligned}$$

Thus the identity map $I : (A, *, |||\cdot|||) \to (A, \cdot, \|\cdot\|)$ is an ϵ-almost multiplicative map. Hence the conclusion follows from Theorem 4.2. $\square$

Proposition 5.7. *Let A be a complex Banach algebra with unit 1 and $0 < \epsilon < \frac{\sqrt{5}-1}{2}$. Let $*$ be an ϵ-perturbation with the same unit, then $\sigma^*(a) \subseteq \sigma_{\epsilon(1+\epsilon)}(a)$*

Proof. Since the multiplication $\cdot$ is an algebraic $\epsilon(1+\epsilon)$-perturbation on $(A, *, |||\cdot|||)$, the result follows from Proposition 5.6. $\square$

Next we obtain some interesting relations connecting condition spectrum of elements in $(A, \cdot, \|\cdot\|)$ and $(A, *, |||\cdot|||)$. The following results will be useful for this purpose.

Lemma 5.8. *Let A be a complex Banach algebra with unit 1 and $0 < \epsilon < 1$. Let $*$ be an ϵ-perturbation with the same unit. If $\lambda \notin \sigma^*_\epsilon(a)$, then $\lambda - a$ is invertible in $(A, \cdot, \|\cdot\|)$ and*

$$\|(\lambda - a)^{-1}\| \leq \frac{\|(\lambda - a)_*^{-1}\|}{1 - \epsilon \|\lambda - a\| \|(\lambda - a)_*^{-1}\|}.$$

Proof. Let $\lambda \notin \sigma^*_\epsilon(a)$. By proposition 5.6, $\lambda - a$ is invertible in $(A, \cdot, \|\cdot\|)$. Denote $b = \lambda - a$. Then

$$\begin{aligned} \|1 - b \cdot b_*^{-1}\| &= \|b * b_*^{-1} - b \cdot b_*^{-1}\| \\ &\leq \epsilon \|b\| \|b_*^{-1}\| \\ &\leq \epsilon |||b||| \ |||b_*^{-1}||| \\ &< 1 \,. \end{aligned}$$

Hence $b \cdot b_*^{-1}$ is invertible in $(A, \cdot, \|\cdot\|)$ and

$$\begin{aligned} \|(b \cdot b_*^{-1})^{-1}\| &\leq \frac{1}{1 - \|1 - b \cdot b_*^{-1}\|} \\ &\leq \frac{1}{1 - \epsilon \|b\| \|b_*^{-1}\|} \,. \end{aligned}$$

Hence

$$\begin{aligned} \|b^{-1}\| &= \|b_*^{-1} \cdot (b \cdot b_*^{-1})^{-1}\| \\ &\leq \|b_*^{-1}\| \|(b \cdot b_*^{-1})^{-1}\| \\ &\leq \frac{\|b_*^{-1}\|}{1 - \epsilon \|b\| \|b_*^{-1}\|} \,. \end{aligned}$$

$\square$

Theorem 5.9. *Let A be a complex Banach algebra with unit 1 and $0 < \epsilon < 1$. Let $*$ be an ϵ-perturbation with the same unit. Then $\sigma_{(t-1)\epsilon}(a) \subseteq \sigma^*_{t\epsilon}(a)$ for all t satisfying $1 < t < \dfrac{1}{\epsilon}$. In particular $\sigma_\epsilon(a) \subseteq \sigma^*_{2\epsilon}(a)$*

Proof. Let $1 < t < \dfrac{1}{\epsilon}$ and $\lambda \notin \sigma^*_{t\epsilon}(a) \supseteq \sigma^*_{\epsilon}(a)$. Then by Lemma 5.8,

$$\|(\lambda - a)^{-1}\| \leq \frac{\|(\lambda - a)_*^{-1}\|}{1 - \epsilon\|\lambda - a\|\|(\lambda - a)_*^{-1}\|}.$$

Hence

$$\begin{aligned} \|\lambda - a\|\|(\lambda - a)^{-1}\| &\leq \frac{\|\lambda - a\|\|(\lambda - a)_*^{-1}\|}{1 - \epsilon\|\lambda - a\|\|(\lambda - a)_*^{-1}\|} \\ &\leq \frac{|||\lambda - a|||\ |||(\lambda - a)_*^{-1}|||}{1 - \epsilon|||\lambda - a|||\ |||(\lambda - a)_*^{-1}|||} \\ &< \frac{1}{(t-1)\epsilon}. \end{aligned}$$

□

Corollary 5.10. *Let A be a complex Banach algebra with unit 1 and $0 < \epsilon < \frac{\sqrt{5}-1}{2}$. Let $*$ be an ϵ-perturbation with the same unit. If $\lambda \notin \sigma_{\epsilon(1+\epsilon)}(a)$, then $\lambda - a$ is invertible in $(A, *, |||\cdot|||)$ and*

$$\|(\lambda - a)_*^{-1}\| \leq \frac{\|(\lambda - a)^{-1}\|}{1 - \epsilon(1+\epsilon)^3\|\lambda - a\|\|(\lambda - a)^{-1}\|}.$$

Proof. Let $\lambda \notin \sigma_{\epsilon(1+\epsilon)}(a)$, denote $b = \lambda - a$. If $*$ is an algebraic ϵ-perturbation on $(A, \cdot, \|\cdot\|)$, then the multiplication $\cdot$ is an algebraic $\epsilon(1+\epsilon)$-perturbation on $(A, *, |||\cdot|||)$. Then by Lemma 5.8,

$$|||b_*^{-1}||| \leq \frac{|||b^{-1}|||}{1 - \epsilon(1+\epsilon)|||b|||\ |||b^{-1}|||}.$$

Since $\|x\| \leq |||x||| \leq (1+\epsilon)\|x\| \quad for \ all \ \ x \in A$, we get

$$\|b_*^{-1}\| \leq \frac{\|b^{-1}\|}{1 - \epsilon(1+\epsilon)^3\|b\|\|b^{-1}\|}.$$

□

Corollary 5.11. *Let A be a complex Banach algebra with unit 1 and $0 < \epsilon < 1$. Let $*$ be an ϵ-perturbation on A with the same unit. Then $\sigma^*_{\frac{\epsilon(t-1)}{1+\epsilon}}(a) \subseteq \sigma_{t\epsilon(1+\epsilon)}(a)$ for all $t > 1$. In particular, $\sigma^*_{\frac{\epsilon}{1+\epsilon}}(a) \subseteq \sigma_{2\epsilon(1+\epsilon)}(a)$.*

Proof. Let $t > 1$ and $\lambda \notin \sigma_{t\epsilon(1+\epsilon)}(a) \supseteq \sigma_{\epsilon(1+\epsilon)}(a)$. Then by Corollary 5.10 $\lambda - a$ is invertible in $(A, *, |||\cdot|||)$ and

$$|||(\lambda - a)_*^{-1}||| \leq \frac{|||(\lambda - a)^{-1}|||}{1 - \epsilon(1+\epsilon)|||\lambda - a|||\ |||\lambda - a^{-1}|||}.$$

Hence,

$$
\begin{aligned}
|||\lambda - a||| \ |||(\lambda - a)_*^{-1}||| &\leq \frac{|||\lambda - a||| \ |||(\lambda - a)^{-1}|||}{1 - \epsilon(1+\epsilon)|||\lambda - a||| \ |||\lambda - a^{-1}|||} \\
&\leq \frac{(1+\epsilon)^2\|\lambda - a\| \ \|(\lambda - a)^{-1}\|}{1 - \epsilon(1+\epsilon)^3\|\lambda - a\| \ \|\lambda - a^{-1}\|} \\
&< \frac{1+\epsilon}{(t-1)\epsilon}.
\end{aligned}
$$

□

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