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Integral Equations and Operator Theory

Approximation Numbers of Operators on Normed Linear Spaces

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Abstract. In [1], Böttcher et. al. showed that if T is a bounded linear operator on a separable Hilbert space H, $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of H and P_n is the orthogonal projection onto the span of $\{e_j\}_{j=1}^{n}$, then for each $k \in \mathbb{N}$, the sequence $\{s_k(P_nTP_n)\}$ converges to $s_k(T)$, where for a bounded operator A on H, $s_k(A)$ denotes the kth *approximation number* of A, that is, $s_k(A)$ is the distance from A to the set of all bounded linear operators of rank at most $k-1$. In this paper we extend the above result to more general cases. In particular, we prove that if T is a bounded linear operator from a separable normed linear space X to a reflexive Banach space Y and if $\{P_n\}$ and $\{Q_n\}$ are sequences of bounded linear operators on X and Y, respectively, such that $\|P_n\|\,\|Q_n\| \leq 1$ for all $n \in \mathbb{N}$ and $\{Q_nTP_n\}$ converges to T under the weak operator topology, then $\{s_k(Q_nTP_n)\}$ converges to $s_k(T)$. We also obtain a similar result for the case of any normed linear space Y which is the dual of some separable normed linear space. For compact operators, we give this convergence of $s_k(Q_nTP_n)$ to $s_k(T)$ with separability assumptions on X and the dual of Y. Counter examples are given to show that the results do not hold if additional assumptions on the space Y are removed. Under separability assumptions on X and Y, we also show that if there exist sequences of bounded linear operators $\{P_n\}$ and $\{Q_n\}$ on X and Y respectively such that (i) Q_nTP_n is compact, (ii) $\|P_n\|\,\|Q_n\| \leq 1$ and (iii) $\{Q_nTP_n\}$ converges to T in the weak operator topology, then $\{s_k(Q_nTP_n)\}$ converges to $s_k(T)$ if and only if $s_k(T) = s_k(T')$. This leads to a generalization of a result of Hutton [3], proved for compact operators between normed linear spaces.

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1. Introduction

Let X and Y be normed linear spaces and $BL(X,Y)$ denote the space of all bounded linear operators from X to Y. We use the notation $BL(X)$ for $BL(X,X)$. The concept of approximation numbers of bounded linear operators in $BL(X,Y)$ is a generalization of the concept of singular values of compact operators between Hilbert spaces. More precisely, for $k \in \mathbb{N}$ and $T \in BL(X,Y)$, the ***k*th approximation number** of T, denoted by $s_k(T)$, is defined as

$$s_k(T) := \inf\{\|T - F\| : F \in BL(X,Y),\ \mathrm{rank}(F) \leq k-1\}.$$

It is obvious that $s_1(T) = \|T\|$ and $s_1(T) \geq s_2(T) \geq \ldots \geq 0$.

Some studies about approximation numbers and their properties can be found in Pietsch [6, 7]. Properties of approximation numbers are found to be useful in estimating errors while solving operator equations (cf. Schock [8]). So it is natural to ask the following general question:

> Suppose $\{T_n\}$ is an approximation of $T \in BL(X,Y)$ in some sense. Under what additional assumptions can one guarantee the convergence $s_k(T_n) \to s_k(T)$ as $n \to \infty$, for each $k \in \mathbb{N}$?

This question has an obvious affirmative answer if $\{T_n\}$ converges to T with respect to the operator norm, that is, if $\|T_n - T\| \to 0$ as $n \to \infty$. Hence, the above question is relevant if other weaker forms of convergence are considered.

In this regard, Böttcher and Grudsky [2] have shown that for a Toeplitz operator T in $BL(\ell^2)$, if P_n is the orthogonal projection onto the space spanned by the first n elements of the standard orthonormal basis of ℓ^2 and $T_n := P_n T P_n$, then for each $k \in \mathbb{N}$, $s_k(T_n) \to s_k(T)$ as $n \to \infty$. Recently, Böttcher, Chithra and Namboodiri [1] have extended the above result in [2] to bounded linear operators in $BL(H)$, where H is a separable complex Hilbert space, as follows.

Theorem 1.1. (cf. [1], Theorem 1.1) *Let H be a separable complex Hilbert space, $T \in BL(H)$ and P_n be the orthogonal projection onto the span of $\{e_j\}_{j=1}^{n}$, where $\{e_j\}_{j=1}^{\infty}$ denotes an orthonormal basis of H. Let $T_n := P_n T P_n$. Then for each $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

The operators T_n in Theorem 1.1 are called the *truncations* of T. It is to be observed that $T_n \to T$ strongly as $n \to \infty$, that is, $T_n x \to Tx$ as $n \to \infty$ for all $x \in X$.

The following lemma is the main ingredient for the proof of Theorem 1.1.

Lemma 1.2. (cf. [1], Lemma 1.2) *Let H be a separable complex Hilbert space. Fix $k \in \mathbb{N}$. Let $\{F_n\}$ be a uniformly bounded sequence of operators in $BL(H)$ such that* $\mathrm{rank}(F_n) \leq k$ *for all $n \in \mathbb{N}$. Then there exists an operator $F \in BL(H)$ with* $\mathrm{rank}(F) \leq k$ *such that for each $x, y \in H$, the sequence $\{\langle y, F_n x\rangle\}$ has a subsequence which converges to $\langle y, Fx\rangle$ as $n \to \infty$.*

Observing the proof of Theorem 1.1 given in [1], we can reformulate Theorem 1.1 in the following general form.

Theorem 1.3. *Let H_1 and H_2 be separable Hilbert spaces and $T \in BL(H_1, H_2)$. Let $\{P_n\}$ and $\{Q_n\}$ be sequences of projections in $BL(H_1)$ and $BL(H_2)$ respectively such that $\|P_n\| = 1 = \|Q_n\|$ for all $n \in \mathbb{N}$, and $P_n x \to x$ and $Q_n y \to y$ for all $x \in H_1$ and $y \in H_2$ as $n \to \infty$. Then $s_k(Q_n T P_n) \to s_k(T)$ as $n \to \infty$.*

The main purpose of this paper is to generalize Theorem 1.3 to the case when Hilbert spaces H_1 and H_2 are replaced by normed linear spaces X and Y, respectively, where X is a separable normed linear space, and Y is either a reflexive Banach space or it is a dual of a separable normed linear space. In the special case of T being a compact operator, we get the conclusion under separability assumptions on X and Y'.

So, let X and Y be normed linear spaces and $T \in BL(X, Y)$. Let $\{P_n\}$ and $\{Q_n\}$ be operators in $BL(X)$ and $BL(Y)$, respectively such that $\|P_n\|\,\|Q_n\| \le 1$ for all $n \in \mathbb{N}$. In Section 2, we consider the case when Y is reflexive and X is separable. For this purpose we generalize Lemma 1.2 with $BL(X, Y)$ in place of $BL(H)$, which holds if and only if Y is a reflexive Banach space. We also show that the infimum in the definition of $s_k(T)$ is attained at a finite rank operator of rank at most $k-1$. This also leads to the conclusion that $s_k(T) = 0$ if and only if $T \in BL(X, Y)$ is of rank at most $k-1$, if Y is a reflexive space and X is separable.

In Section 3, we extend the results in Section 2 to the case in which Y is only assumed to be a dual of a separable normed linear space. The main theorem of Section 4 includes a generalization of Theorem 1.3 for compact operators T under separability assumptions on the dual space of Y and either reflexivity or separability assumptions on X.

In Section 4 we also address the question whether $s_k(T) = s_k(T')$ for all $k \in \mathbb{N}$ and $T \in BL(X, Y)$, which has been answered affirmatively by Hutton [3] if T is a compact operator. It is also shown in [3], using a counter example, that the equality $s_k(T) = s_k(T')$ need not hold if T is not a compact operator. The main theorem of Section 4 leads to an extension of the above referred result of Hutton [3] for a class of operators in $BL(X, Y)$ which can be approximated by certain compact operators, with some additional assumptions on the spaces. It is also shown that the convergence of $\{s_k(T_n)\}$ to $s_k(T)$ is closely related to the equality $s_k(T) = s_k(T')$.

For our results, we shall make use of the following two definitions.

Definition 1.4. We say that a sequence $\{T_n\}$ of operators in $BL(X, Y)$ **converges** to $T \in BL(X, Y)$ **in the weak operator topology** if for all $x \in X$, $T_n x \xrightarrow{w} Tx$; that is, for every $x \in X$, $f \in Y'$, $f(T_n x) \to f(Tx)$ as $n \to \infty$. We denote this fact as $T_n \xrightarrow{wo} T$.

Definition 1.5. We say that a sequence of operators $\{A_n\}$ in $BL(X, Z')$, where X and Z are normed linear spaces, **converges** to $A \in BL(X, Z')$ **in the weak***

operator topology if for every $x \in X$, $A_n x \xrightarrow{w^*} Ax$; that is, for every $x \in X$ and $z \in Z$, $A_n x(z) \to Ax(z)$ as $n \to \infty$. We denote this fact as $A_n \xrightarrow{wo^*} A$.

It can be seen easily that the strong convergence of operators implies the convergence in the weak operator topology, and if the codomain is the dual of a normed linear space, then convergence in the weak operator topology implies convergence in the weak* operator topology.

2. Approximation under the reflexivity assumption

In this section we generalize Theorem 1.3 for operators in $BL(X, Y)$, where X is separable and Y is reflexive. As a first step towards that we generalize Lemma 1.2. For this purpose we prove the following three results. The first one is given as an exercise in Limaye [4]. We give its proof here for the sake of completeness.

Lemma 2.1. (cf. [4], Exercise 5-9) *Let X_0 be a k-dimensional normed linear space. Then there exists a basis $E = \{a_1, a_2, \ldots, a_k\}$ for X_0 such that $\|a_i\| = 1$ and $dist(a_i, Y_i) = 1$, where $Y_i = span\{E \setminus \{a_i\}\}$, for all $i = 1, 2, \ldots, k$.*

Proof. Let B denote the closed unit ball of X_0 and put $B^k := B \times B \times \cdots \times B$ (k terms). Let $\{y_1, y_2, \ldots, y_k\}$ be a basis for X_0 such that $\|y_i\| = 1$ for all $i = 1, 2, \ldots, k$. We define the map $det : B^k \to \mathbb{K}$ by

$$det(x_1, x_2, \ldots, x_k) = \det[\beta_{ij}]_{k\times k}, \quad (x_1, x_2, \ldots, x_k) \in B^k,$$

where for each $i \in \{1, \ldots, k\}$, $(\beta_{i1}, \ldots, \beta_{ik})$ is the unique k-tuple of complex numbers such that $x_i = \sum_{j=1}^{k} \beta_{ij} y_j$, and $\det[\beta_{ij}]_{k\times k}$ denotes the determinant of the $k \times k$ matrix $[\beta_{ij}]_{k\times k}$. We observe that, for $(x_1, x_2, \ldots, x_k) \in B^k$, $\{x_1, x_2, \ldots, x_k\}$ is linearly independent if and only if $det(x_1, x_2, \ldots, x_k) > 0$.

Since B^k is compact and det is a continuous function, it attains its maximal value in B^k, say at $a = (a_1, a_2, \ldots, a_k) \in B^k$. Since $a \in B^k$, $\|a_i\| \leq 1$ for all $i = 1, 2, \ldots, k$. Since $det(y_1, y_2, \ldots, y_k) = \det[\delta_{ij}] = 1$, we have $det(a) \geq 1$. This shows that the set $E = \{a_1, a_2, \ldots, a_k\}$ is linearly independent and is a basis of X_0. Now for $i \in \{1, 2, \ldots, k\}$, let $b^{(i)} = (b_1^{(i)}, b_2^{(i)}, \ldots, b_k^{(i)})$ with

$$b_j^{(i)} := \begin{cases} a_j, & \text{if } j \neq i, \\ \dfrac{a_i}{\|a_i\|}, & \text{if } j = i. \end{cases}$$

Then $b^{(i)} \in B^k$ and hence $det(a) \geq det(b^{(i)}) = det(a)/\|a_i\|$. Thus we also have $\|a_i\| \geq 1$ so that $\|a_i\| = 1$ for all $i = 1, 2, \ldots, k$.

Now let $Y_i = \text{span}\{E \setminus \{a_i\}\}$ and $z \in Y_i$ for $i = 1, 2, \ldots, k$. Since $a_i \notin Y_i$, we have $\|a_i - z\| > 0$ and $\text{dist}(a_i, Y_i) \leq 1$. Now for $i = 1, 2, \ldots, k$, let $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, \ldots, u_k^{(i)})$ with

$$u_j^{(i)} := \begin{cases} a_j, & \text{if } j \neq i, \\ \dfrac{a_i - z}{\|a_i - z\|}, & \text{if } j = i. \end{cases}$$

Then $u^{(i)} \in B^k$ and so $det(a) \geq det(u^{(i)}) = det(a)/\|a_i - z\|$. Hence $\|a_i - z\| \geq 1$ for all $z \in Y_i$, which gives $\text{dist}(a_i, Y_i) = 1$. □

Lemma 2.2. *Let X_0 be a k-dimensional subspace of a normed linear space X. Then there exist a basis $E = \{a_1, a_2, \dots, a_k\}$ for X_0 and a set $\{f_1, f_2, \dots, f_k\} \subseteq X'$ such that $\|a_i\| = 1 = \|f_i\|$ and $f_i(a_j) = \delta_{ij}$ for all $i, j \in \{1, 2, \dots, k\}$.*

Proof. From Lemma 2.1, there exists a basis $E = \{a_1, a_2, \dots, a_k\}$ for X_0 such that $\|a_i\| = 1$ and $\text{dist}(a_i, Y_i) = 1$ for all $i = 1, 2, \dots, k$, where $Y_i = \text{span}\{E \setminus \{a_i\}\}$. Since Y_i is closed and $a_i \notin Y_i$, by a consequence of the Hahn-Banach theorem (cf. Nair [5], Corollary 5.5), there exists a linear functional $f_i \in X'$ such that

$$f_i|_{Y_i} = 0, \quad \|f_i\| = 1 \quad \text{and} \quad \mathrm{f_i(a_i) = dist(a_i, Y_i)} = 1.$$

For $i \neq j$, $a_j \in Y_i$ and so $f_i(a_j) = 0$. This proves that $f_i(a_j) = \delta_{ij}$ for all $i, j \in \{1, 2, \dots, k\}$. □

Proposition 2.3. *Let X and Y be normed linear spaces and $k \in \mathbb{N}$. Then corresponding to any $T \in BL(X, Y)$ of rank k, there exist a basis $\{a_1, a_2, \dots, a_k\}$ for $R(T)$ with $\|a_j\| = 1$ and a set $\{\psi_1, \psi_2, \dots, \psi_k\} \subseteq X'$ with $\|\psi_j\| \leq \|T\|$, $j = 1, 2, \dots, k$, such that for all $x \in X$, $Tx = \sum_{j=1}^k \psi_j(x) a_j$.*

Proof. By Lemma 2.2, there exist a basis $\{a_1, a_2, \dots, a_k\}$ for $R(T)$ and a set $\{f_1, f_2, \dots, f_k\} \subseteq Y'$ such that $\|a_j\| = 1 = \|f_j\|$ and $f_i(a_j) = \delta_{ij}$ for all $i, j \in \{1, 2, \dots, k\}$. Then it follows that $y = \sum_{j=1}^k f_j(y) a_j$ for all $y \in R(T)$. In particular, $Tx = \sum_{j=1}^k f_j(Tx) a_j$ for all $x \in X$. Define $\psi_j : X \to \mathbb{C}$ by $\psi_j(x) = f_j(Tx)$ for $i = 1, 2, \dots, k$. Then $\psi_j \in X'$ and

$$\|\psi_j\| = \|f_j \circ T\| \leq \|f_j\| \, \|T\| = \|T\| \quad \forall\, j = 1, 2, \dots, k$$

and $Tx = \sum_{j=1}^k \psi_j(x) a_j$. □

Now, we give the result which generalizes Lemma 1.2.

Lemma 2.4. *Let X be a separable normed linear space and Y be a reflexive Banach space. Let $k \in \mathbb{N}$ and $\{T_n\}$ be a uniformly bounded sequence of operators in $BL(X, Y)$ with $\text{rank}(T_n) \leq k$ for all $n \in \mathbb{N}$. Then there exist an operator $T \in BL(X, Y)$ with $\text{rank}(T) \leq k$ and a subsequence $\{T_{n_\ell}\}$ of $\{T_n\}$ such that $T_{n_\ell} \xrightarrow{wo} T$ as $\ell \to \infty$.*

Proof. Let $M > 0$ be such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. Since $k_n := \text{rank}(T_n) \leq k$, by Proposition 2.3, we can find $a_1^{(n)}, a_2^{(n)}, \dots, a_{k_n}^{(n)}$ in $R(T_n)$ and $\psi_1^{(n)}, \psi_2^{(n)}, \dots, \psi_{k_n}^{(n)}$ in X' such that $\|a_j^{(n)}\| = 1$ and $\|\psi_j^{(n)}\| \leq \|T_n\| \leq M$ for $j = 1, 2, \dots, k_n$ and for all $n \in \mathbb{N}$. Also, for $x \in X$, $\quad T_n x = \sum_{j=1}^{k_n} \psi_j^{(n)}(x) a_j^{(n)}$ for all $n \in \mathbb{N}$. If $k_n < k$ for some n, then taking $a_j^{(n)} = 0$ and $\psi_j^{(n)} = 0$ for $j > k_n$, we can write

$$T_n x = \sum_{j=1}^k \psi_j^{(n)}(x) a_j^{(n)}, \quad x \in X,\, n \in \mathbb{N}.$$

Since Y is reflexive, by the Eberlein-Shmulyan theorem (cf. [5], Theorem 8.25), for each j, the bounded sequence $\{a_j^{(n)}\}$ has a weakly convergent subsequence. Since X is separable, we also know that for each j, the bounded sequence $\{\psi_j^{(n)}\}$ has a weak* convergent subsequence (cf. [4], Theorem 15.4). Thus, it follows by considering subsequences that there exist $a_1, a_2, \ldots, a_k$ in Y and $\psi_1, \psi_2, \ldots, \psi_k$ in X' such that $\|a_j\| \leq 1,\ \|\psi_j\| \leq M$ and a strictly increasing sequence $\{n_\ell\}$ in $\mathbb{N}$ such that

$$a_j^{(n_\ell)} \xrightarrow{w} a_j, \quad \psi_j^{(n_\ell)} \xrightarrow{w^*} \psi_j \quad \text{as} \quad \ell \to \infty$$

for $j = 1, 2, \ldots, k$. Define $T : X \to Y$ by

$$Tx = \sum_{j=1}^{k} \psi_j(x) a_j, \quad x \in X.$$

Then it follows that $T \in BL(X, Y)$, $\text{rank}(T) \leq k$ and for each $x \in X$ and $f \in Y'$,

$$f(T_{n_\ell} x) = \sum_{j=1}^{k} \psi_j^{(n_\ell)}(x) f(a_j^{(n_\ell)}) \to \sum_{j=1}^{k} \psi_j(x) f(a_j) = f(Tx) \quad \text{as} \quad \ell \to \infty. \qquad \square$$

As a corollary to the above theorem we prove that for $T \in BL(X, Y)$, where X is separable and Y is reflexive, $s_k(T)$ is attained at some finite rank operator $F \in BL(X, Y)$ of rank atmost $k - 1$.

Corollary 2.5. *Let X and Y be as in Lemma* 2.4*, $T \in BL(X, Y)$ and $k \in \mathbb{N}$. Then there exists an operator $F \in BL(X, Y)$ with* $\text{rank}(F) \leq k - 1$ *such that $\|T - F\| = s_k(T)$. In particular, $s_k(T) = 0$ if and only if* $\text{rank}(T) \leq k - 1$.

Proof. Let $s_k(T) = d$. For each $n \in \mathbb{N}$, there exist $F_n \in BL(X, Y)$ such that $\text{rank}(F_n) \leq k - 1$ and $\|T - F_n\| < d + \frac{1}{n}$. Thus $\|F_n\| \leq \|T\| + d + 1$ for all $n \in \mathbb{N}$. Hence by Lemma 2.4, there exist an operator $F \in BL(X, Y)$ with $\text{rank}(F) \leq k - 1$ and a subsequence $\{F_{n_j}\}$ of $\{F_n\}$ such that $F_{n_j} \xrightarrow{wo} F$ as $j \to \infty$.

Now let $\epsilon > 0$, $x \in X$, $f \in Y'$ with $\|x\| \leq 1$, $\|f\| \leq 1$. Then there exists an $n_j \in \mathbb{N}$ such that $\frac{1}{n_j} < \frac{\epsilon}{2}$ and $|f(Fx) - f(F_{n_j} x)| < \frac{\epsilon}{2}$. Then

$$\begin{aligned} |f(Tx) - f(Fx)| &\leq |f(Tx) - f(F_{n_j} x)| + |f(F_{n_j} x) - f(Fx)| \\ &\leq \|T - F_{n_j}\| + \frac{\epsilon}{2} < d + \frac{1}{n_j} + \frac{\epsilon}{2} < d + \epsilon. \end{aligned}$$

Since this holds for all $x \in X$, $f \in Y'$ with $\|x\| \leq 1$, $\|f\| \leq 1$ and $\epsilon > 0$ is arbitrary, we get $\|T - F\| \leq d$. On the other hand, since $\text{rank}(F) \leq k - 1$, we have $\|T - F\| \geq d$. Hence $\|T - F\| = d$. From this, the particular case is obvious. $\square$

The following example shows that the conclusion of the Lemma 2.4 does not hold if the space Y is not reflexive.

Example. Let $T_n : \ell^1 \to \ell^1$ be defined by

$$T_n x = x(1) e_n, \quad x = (x(1), x(2), \ldots) \in \ell^1, \quad n \in \mathbb{N}.$$

Clearly $T_n \in BL(\ell^1)$ with $\operatorname{rank}(T_n) = 1$ and $\|T_n\| = 1$ for all $n \in \mathbb{N}$. We claim that $\{T_n\}$ does not have a subsequence which converges to an operator with respect to the weak operator topology. To see this, suppose there exist a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and an operator $T \in BL(\ell^1)$ such that $T_{n_k}x \xrightarrow{w} Tx$ for each $x \in \ell^1$ as $k \to \infty$. Then $f(T_{n_k}x) \to f(Tx)$ as $k \to \infty$ for all $f \in (\ell^1)'$ and $x \in \ell^1$. In particular, $f(e_{n_k}) = f(T_{n_k}e_1) \to f(Te_1)$ as $k \to \infty$ holds also for $f \in (\ell^1)'$ defined by

$$f(x) = \sum_{k=1}^{\infty} (-1)^k x(n_k), \quad x \in \ell^1.$$

But then $f(e_{n_k}) = (-1)^k$, giving a contradiction to the convergence of $f(e_{n_k})$ and hence to the existence of such a subsequence $\{T_{n_k}\}$.

Remark 2.6. Note that the sequence $\{T_n\}$ in the preceding example converges to 0 in the weak* operator topology if ℓ^1 is regarded as the dual space of c_{00} or c_0 with respect to the norm $\|\cdot\|_\infty$. To see this, let J denote the canonical isometry from $\ell^1 \to (c_0)'$, defined by

$$(Jx)(y) = \sum_{j=1}^{\infty} x(j)y(j), \quad x \in \ell^1,\, y \in c_0.$$

Then $JT_nJ^{-1} \in BL((c_0)')$. Now, for $g \in (c_0)'$, let $y \in \ell^1$ be such that $J(y) = g$. Then for $x \in X$,

$$(JT_nJ^{-1}g)(x) = (JT_ny)(x) = y(1)(Je_n)(x) = y(1)x(n) \to 0 \quad \text{as } n \to \infty.$$

Thus $JT_nJ^{-1} \xrightarrow{wo^*} 0$ as $n \to \infty$. It can be seen that $JT_nJ^{-1} \not\xrightarrow{wo} 0$ as $n \to \infty$.

The following proposition shows that, for Lemma 2.4 to hold, reflexivity of Y is not only sufficient but also necessary.

Proposition 2.7. *If Y is a non-reflexive space, then there exists a uniformly bounded sequence $\{T_n\}$ of operators in $BL(X,Y)$ such that* $\operatorname{rank}(T_n) \le 1$ *for all $n \in \mathbb{N}$ and $\{T_n\}$ does not have any subsequence which converges in the weak operator topology.*

Proof. Suppose Y is not reflexive. Then, by Eberlein's theorem (cf. [4], Theorem 16.5), there exists a bounded sequence $\{u_n\}$ in Y which does not have a weakly convergent subsequence.

Let $a \in X$ with $\|a\| = 1$. By a consequence of the Hahn-Banach Extension Theorem, there exists a functional $g \in X'$ of norm 1 such that $g(a) = \|a\| = 1$ ([5], Corollary 5.6). We define $T_n : X \to Y$ by $T_nx = g(x)u_n$ for $x \in X$. Then for each $n \in \mathbb{N}$, $T_n \in BL(X,Y)$ is of rank 1 and the sequence $\{\|T_n\|\}$ is bounded. We claim that $\{T_n\}$ does not have any subsequence which converges in the weak operator topology.

Suppose there exist a subsequence $\{T_{n_j}\}$ of $\{T_n\}$ and an operator $T \in BL(X,Y)$ such that $T_{n_j}x \xrightarrow{w} Tx$ as $j \to \infty$, for all $x \in X$. This gives $f(u_{n_j}) = f(T_{n_j}a) \to f(Ta)$ as $j \to \infty$, for every $f \in Y'$. But this gives a weakly convergent subsequence of $\{u_n\}$, contradicting the choice of $\{u_n\}$. $\square$

Now, we prove the main theorem of this section which generalizes Theorem 1.3 to operators in $BL(X,Y)$, when X is separable and Y is reflexive.

Theorem 2.8. *Let X and Y be normed linear spaces, $T \in BL(X,Y)$, and $\{P_n\}$ and $\{Q_n\}$ be sequences of operators in $BL(X)$ and $BL(Y)$ respectively such that $\|P_n\|\,\|Q_n\| \le 1$ for all $n \in \mathbb{N}$. Then $s_k(Q_nTP_n) \le s_k(T)$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Further, if X is separable, Y is reflexive and $T_n := Q_nTP_n \xrightarrow{wo} T$ as $n \to \infty$, then for each $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} s_k(T_n) = s_k(T).$$

Proof. Fix $k \in \mathbb{N}$ and denote $d := s_k(T)$ and $d_n := s_k(T_n)$. For $\epsilon > 0$, let $F \in BL(X,Y)$ be such that $\text{rank}(F) \le k-1$ and $\|T - F\| < d+\epsilon$. Then for any $n \in \mathbb{N}$,

$$\|Q_nTP_n - Q_nFP_n\| \le \|Q_n\|\,\|T-F\|\,\|P_n\| \le \|T-F\| < d+\epsilon$$

and $\text{rank}(Q_nFP_n) \le k-1$. Hence $d_n < d+\epsilon$ for all $n \in \mathbb{N}$. Since $\epsilon > 0$ was arbitrary, we obtain $\sup_n s_k(T_n) \le s_k(T)$. This proves the first part of the conclusion.

Next, let X be separable, Y be reflexive and $T_n := Q_nTP_n \xrightarrow{wo} T$ as $n \to \infty$. Then the conclusion holds trivially if $d = 0$. Assume $d > 0$ and $d_n \not\to d$. Then there exists an $\epsilon > 0$ such that $d_n < d - \epsilon$ for infinitely many n. Hence there exist operators $F_{n_j} \in BL(X,Y)$ such that $\text{rank}(F_{n_j}) \le k-1$ and $\|Q_{n_j}TP_{n_j} - F_{n_j}\| < d - \epsilon$ for all $j \in \mathbb{N}$. Thus

$$\|F_{n_j}\| \le \|F_{n_j} - Q_{n_j}TP_{n_j}\| + \|Q_{n_j}TP_{n_j}\| < d + \|T\| \quad \forall j \in \mathbb{N}.$$

Hence by Lemma 2.4, there exist an operator $F \in BL(X,Y)$ with $\text{rank}(F) \le k-1$ and a subsequence $\{F_{n_{j_\ell}}\}$ of $\{F_{n_j}\}$ such that for each $x \in X$ and $f \in Y'$, $|f(Fx) - f(F_{n_{j_\ell}}x)| \to 0$ as $\ell \to \infty$.

Now let $x \in X, f \in Y'$ be such that $\|x\| \le 1$ and $\|f\| \le 1$. Then

$$\begin{aligned}|f(Tx) - f(Fx)| \le &\; |f(Tx) - f(Q_{n_{j_\ell}}TP_{n_{j_\ell}}x)| + |f(Q_{n_{j_\ell}}TP_{n_{j_\ell}}x) - f(F_{n_{j_\ell}}x)| \\ &+ |f(F_{n_{j_\ell}}x) - f(Fx)|\end{aligned}$$

Note that for each ℓ,

$$|f(Q_{n_{j_\ell}}TP_{n_{j_\ell}}x) - f(F_{n_{j_\ell}}x)| \le \|Q_{n_{j_\ell}}TP_{n_{j_\ell}} - F_{n_{j_\ell}}\| < d - \epsilon,$$

whereas the terms $|f(Tx) - f(Q_{n_{j_\ell}}TP_{n_{j_\ell}}x)|$ and $|f(F_{n_{j_\ell}}x) - f(Fx)|$ can be made less than $\epsilon/3$ by choosing ℓ sufficiently large. Hence $|f(Tx) - f(Fx)| \le d - \epsilon/3$. Since this holds for each $x \in X$ and $f \in Y'$ with $\|x\| \le 1$ and $\|f\| \le 1$, we have $\|T - F\| \le d - \epsilon/3$ so that $d \le d - \epsilon/3$. Thus we arrive at a contradiction. Hence $d_n \to d$ as $n \to \infty$. □

Corollary 2.9. *Let X and Y be as in Lemma 2.4 and $T \in BL(X,Y)$. Let $\{P_n\}$ and $\{Q_n\}$ be as in Theorem 2.8. Further, if $P_nx \to x$ and $Q_ny \xrightarrow{w} y$ as $n \to \infty$ for every $x \in X$ and $y \in Y$, then $s_k(T_n) \to s_k(T)$ as $n \to \infty$, for each $k \in \mathbb{N}$.*

Proof. For $x \in X$ and $f \in Y'$,

$$|f(Q_nTP_nx) - f(Tx)| \le |f(Q_nTP_nx) - f(Q_nTx)| + |f(Q_nTx) - f(Tx)| \\ \le \|f\| \, \|Q_n\| \, \|T\| \, \|P_nx - x\| + |f(Q_n(Tx)) - f(Tx)|,$$

which tends to 0 as $n \to \infty$, since $P_nx \to x$ and $f(Q_ny) \to f(y)$ as $n \to \infty$, for every $x \in X$, $y \in Y$ and $f \in Y'$. This implies $T_n \xrightarrow{wo} T$ as $n \to \infty$. Hence the result follows from Theorem 2.8. □

In Theorem 2.8, the operators P_n or Q_n need not be projections.

Example. For $n = 2, 3, \ldots$, consider $P_n : \ell^2 \to \ell^2$ defined by

$P_n(x(1), x(2), \ldots) = (x(1), \ldots, x(n-2), x(n), x(n-1), 0, \ldots)$, $(x(1), x(2), \ldots) \in \ell^2$.

Then $P_n \in BL(\ell^2)$ with $\|P_n\| = 1$ and $P_nx \to x$ as $n \to \infty$, for all $x \in X$.

Now for $X = Y = \ell^2$, $P_n = Q_n$ satisfy all the assumptions of Theorem 2.8. It is clear that P_n is not a projection operator on ℓ^2.

3. Approximation under the duality assumption

In view of Proposition 2.7 we know that the conclusion in Lemma 2.4 holds if and only if the codomain Y is reflexive. Now we prove a result, namely Lemma 3.1, analogous to Lemma 2.4 for the case when Y is not necessarily reflexive, but is the dual space of a separable normed linear space. Here the convergence in weak operator topology is also weakened by convergence in the weak* operator topology. The arguments in the proof of Lemma 3.1 and the subsequent corollary are similar to that of Lemma 2.4 and Corollary 2.5 with weak sense of convergence replaced by weak* sense of convergence. However, for the sake of completeness of the exposition, we supply a detailed proof.

Lemma 3.1. *Let X be a separable normed linear space, Y be the dual space of a separable normed linear space. Let $k \in \mathbb{N}$ and $\{T_n\}$ be a uniformly bounded sequence in $BL(X, Y)$ with* $\operatorname{rank}(T_n) \le k$ *for all $n \in \mathbb{N}$. Then there exist an operator $T \in BL(X, Y)$ with* $\operatorname{rank}(T) \le k$ *and a subsequence $\{T_{n_\ell}\}$ of $\{T_n\}$ such that $T_{n_\ell} \xrightarrow{wo^*} T$ as $\ell \to \infty$.*

Proof. Since $\operatorname{rank}(T_n) =: k_n \le k$ for all $n \in \mathbb{N}$, by Proposition 2.3, we can write

$$T_nx = \sum_{i=1}^{k_n} \psi_i^{(n)}(x) w_i^{(n)}, \quad x \in X, \, n \in \mathbb{N},$$

where $\{w_i^{(n)}\}_{i=1}^{k_n}$ forms a basis of range of T_n with $\|w_i^{(n)}\| \le 1$ and $\psi_i^{(n)} \in X'$ with $\|\psi_i^{(n)}\| \le M$, for some $M > 0$ and for all $n \in \mathbb{N}$, $i = 1, 2, \ldots, k$. Here also if $k_n < k$ for some n, then taking $w_j^{(n)} = 0$ and $\psi_j^{(n)} = 0$ for $j > k_n$, we can write

$$T_nx = \sum_{i=1}^{k} \psi_i^{(n)}(x) w_i^{(n)}, \quad x \in X, \, n \in \mathbb{N}.$$

By hypothesis, X is separable and $Y = Z'$, for some separable normed linear space Z. Therefore, for each i, the bounded sequences $\{w_i^{(n)}\}$ in Z' and $\{\psi_i^{(n)}\}$ in X' have weak* convergent subsequences. Thus, it follows by considering subsequences that there exist $w_i \in Y$ and $\psi_i \in X'$, $i = 1, 2, \ldots, k$ such that

$$\psi_i^{(n_\ell)}(x) \to \psi_i(x), \quad w_i^{(n_\ell)}(z) \to w_i(z)$$

as $\ell \to \infty$ for every $x \in X$, $z \in Z$. Define $T : X \to Y$ by

$$Tx = \sum_{i=1}^{k} \psi_i(x) w_i, \quad x \in X.$$

Then $T \in BL(X, Y)$, $\operatorname{rank}(T) \leq k$ and for each $x \in X, z \in Z$, we have

$$(T_{n_\ell} x)(z) = \sum_{j=1}^{k} g_j^{(n_\ell)}(x) w_j^{(n_\ell)}(z) \to \sum_{j=1}^{k} g_j(x) w_j(z) = Tx(z),$$

as $\ell \to \infty$. The above operator T satisfies the requirements in the lemma. □

Corollary 3.2. *Let X and Y be as in Lemma* 3.1*, $T \in BL(X, Y)$ and $k \in \mathbb{N}$. Then there exists an operator $F \in BL(X, Y)$ with* $\operatorname{rank}(F) \leq k - 1$ *such that $\|T - F\| = s_k(T)$. In particular, $s_k(T) = 0$ if and only if* $\operatorname{rank}(T) \leq k - 1$*.*

Proof. Let $s_k(T) = d$. For all $n \in \mathbb{N}$, there exist $F_n \in BL(X, Y)$ such that $\operatorname{rank}(F_n) \leq k - 1$ and $\|T - F_n\| < d + \frac{1}{n}$. Thus $\|F_n\| \leq \|T\| + d + 1$ for all $n \in \mathbb{N}$. Hence by Lemma 3.1, there exist an operator $F \in BL(X, Y)$ with $\operatorname{rank}(F) \leq k-1$ and a subsequence $\{F_{n_j}\}$ of $\{F_n\}$ such that $F_{n_j} \xrightarrow{wo^*} F$ as $j \to \infty$.

Now let $\epsilon > 0$, $x \in X$, $z \in Z$ with $\|x\| \leq 1$, $\|z\| \leq 1$. Then there exists an $n_j \in \mathbb{N}$ such that $\frac{1}{n_j} < \frac{\epsilon}{2}$ and $|(Fx)(z) - (F_{n_j} x)(z)| < \frac{\epsilon}{2}$. Then

$$\begin{aligned} |(Tx)(z) - (Fx)(z)| &\leq |(Tx)(z) - (F_{n_j} x)(z)| + |(F_{n_j} x)(z) - (Fx)(z)| \\ &\leq \|T - F_{n_j}\| + \frac{\epsilon}{2} < d + \frac{1}{n_j} + \frac{\epsilon}{2} < d + \epsilon. \end{aligned}$$

Since this holds for all $x \in X$, $z \in Z$ with $\|x\| \leq 1$, $\|z\| \leq 1$, and $\epsilon > 0$ is arbitrary, we get $\|T - F\| \leq d$. On the other hand since $\operatorname{rank}(F) \leq k - 1$, $\|T - F\| \geq d$. Hence $\|T - F\| = d$.

If $s_k(T) = 0$, then $T = F$ from the above and hence $\operatorname{rank}(T) \leq k - 1$. □

Using Lemma 3.1 we prove a theorem analogous to Theorem 2.8 for operators in $BL(X, Y)$, where X and Y are as in Lemma 3.1.

Theorem 3.3. *Let X and Y be as in Lemma* 3.1 *and $T \in BL(X, Y)$. Let $\{P_n\}$ and $\{Q_n\}$ be sequences of operators in $BL(X)$ and $BL(Y)$ respectively such that $\|P_n\| \, \|Q_n\| \leq 1$ for each $n \in \mathbb{N}$. If $T_n := Q_n T P_n \xrightarrow{wo^*} T$ as $n \to \infty$, then for each $k \in \mathbb{N}$,*

$$\lim_{n \to \infty} s_k(T_n) = s_k(T).$$

Proof. Let $k \in \mathbb{N}$ and denote $d := s_k(T)$ and $d_n := s_k(T_n)$. From Theorem 2.8 we have $\sup_n s_k(T_n) \leq s_k(T)$. The conclusion holds trivially if $d = 0$. So assume $d > 0$ and $d_n \not\to d$. Then there exist an $\epsilon > 0$ and infinitely many n such that $d_n < d - \epsilon$. Hence there exist operators $F_{n_j} \in BL(X, Y)$ with $\text{rank}(F_{n_j}) \leq k - 1$ such that $\|Q_{n_j} T P_{n_j} - F_{n_j}\| < d - \epsilon$ for all $j \in \mathbb{N}$. Thus

$$\|F_{n_j}\| \leq \|F_{n_j} - Q_{n_j} T P_{n_j}\| + \|Q_{n_j} T P_{n_j}\| < d + \|T\| \quad \forall j \in \mathbb{N}.$$

Hence by Lemma 3.1, there exist an operator $F \in BL(X, Y)$ with rank $(F) \leq k-1$ and a subsequence $\{F_{n_{j_\ell}}\}$ of $\{F_{n_j}\}$ such that for each $x \in X$, $z \in Z$, $|(Fx)(z) - (F_{n_{j_\ell}} x)(z)| \to 0$ as $\ell \to \infty$. Now, let $x \in X$, $z \in Z$ be such that $\|x\| \leq 1$ and $\|z\| \leq 1$. Then

$$\begin{aligned} |(Tx)(z) - (Fx)(z)| \leq\ & |(Tx)(z) - (Q_{n_{j_\ell}} T P_{n_{j_\ell}} x)(z)| \\ & + |(Q_{n_{j_\ell}} T P_{n_{j_\ell}} x)(z) - (F_{n_{j_\ell}} x)(z)| \\ & + |(F_{n_{j_\ell}} x)(z) - (Fx)(z)|. \end{aligned}$$

Note that for each ℓ,

$$|(Q_{n_{j_\ell}} T P_{n_{j_\ell}} x)(z) - (F_{n_{j_\ell}} x)(z)| \leq \|Q_{n_{j_\ell}} T P_{n_{j_\ell}} - F_{n_{j_\ell}}\| < d - \epsilon,$$

whereas the terms $|(Tx)(z) - (Q_{n_{j_\ell}} T P_{n_{j_\ell}} x)(z)|$ and $|(F_{n_{j_\ell}} x)(z) - (Fx)(z)|$ can be made less than $\epsilon/3$ by choosing ℓ sufficiently large. Hence $|(Tx)(z) - (Fx)(z)| < d - \epsilon/3$. Since this holds for each $x \in X$ and $z \in Z$ with $\|x\| \leq 1$ and $\|z\| \leq 1$, we have $\|T - F\| \leq d - \epsilon/3$ so that $d \leq d - \epsilon/3$. Thus we arrive at a contradiction. Hence $d_n \to d$ as $n \to \infty$. □

Corollary 3.4. *Let X and Y be as in Lemma* 3.1 *and $T \in BL(X, Y)$. Let $\{P_n\}$ and $\{Q_n\}$ be as in Theorem* 3.3 *such that $P_n x \to x$ and $Q_n y \xrightarrow{w^*} y$ as $n \to \infty$ for all $x \in X$, $y \in Y$. Then for each $k \in \mathbb{N}$,*

$$\lim_{n \to \infty} s_k(T_n) = s_k(T).$$

Proof. Let $x \in X$ and $z \in Z$. Then

$$\begin{aligned} & |(Q_n T P_n x)(z) - (Tx)(z)| \\ & \leq |(Q_n T P_n x)(z) - (Q_n T x)(z)| + |(Q_n T x)(z) - (Tx)(z)| \\ & \leq \|Q_n\| \, \|T\| \, \|P_n x - x\| \, \|z\| + |(Q_n(Tx))(z) - (Tx)(z)|, \end{aligned}$$

which tends to 0 as $n \to \infty$, since $P_n x \to x$ and $(Q_n y)(z) \to y(z)$ as $n \to \infty$, for every $x \in X$, $y \in Y$ and $z \in Z$. This implies $T_n \xrightarrow{wo^*} T$ as $n \to \infty$. Hence the result follows from Theorem 3.3. □

Remark 3.5. We observe that the conclusion in Theorem 3.3 follows even if we take Y to be linearly isometric with the dual of a separable space Z and assume $JT_n \xrightarrow{wo^*} JT$ as $n \to \infty$, where J denotes the linear isometry from Y to Z'.

Further, we observe that if $\{T_n\}$ is a sequence in $BL(X,Y)$ and $T \in BL(X,Y)$, then $T_n \xrightarrow{wo} T$ if and only if $T'_n \xrightarrow{wo^*} T'$. This is seen as follows:

$$\begin{aligned} T_n \xrightarrow{wo} T \quad &\iff \quad f(T_n x) \to f(Tx) && \forall f \in Y',\, x \in X \\ &\iff \quad (T'_n f)(x) \to (T'f)(x) && \forall\, f \in Y',\, x \in X \\ &\iff \quad T'_n f \xrightarrow{w^*} T'f && \forall\, f \in Y' \\ &\iff \quad T'_n \xrightarrow{wo^*} T' && \text{as } n \to \infty \end{aligned}$$

Also, if X is reflexive, then $T_n \to T$ in the strong sense implies $T'_n \xrightarrow{wo} T'$. To see this, let J denote the canonical isometry from X onto X'' and $\phi \in X''$. Let $x \in X$ be such that $Jx = \phi$. Then for every $f \in Y'$,

$$\begin{aligned} |\phi(T'_n f) - \phi(T'f)| &= |(Jx)(T'_n f) - (Jx)(T'f)| = |(T'_n f)(x) - (T'f)(x)| \\ &= |f(T_n x) - f(Tx)| \le \|f\|\,\|T_n x - Tx\|. \end{aligned}$$

Hence, if $\|T_n x - Tx\| \to 0$ as $n \to \infty$ for every $x \in X$, then $T'_n \xrightarrow{wo} T'$ as $n \to \infty$.

In view of Remark 3.5 together with Theorems 3.3 and 2.8, we obtain the following corollary.

Corollary 3.6. *Let $T \in BL(X,Y)$ and $T_n := Q_n T P_n$ for all $n \in \mathbb{N}$, where $\{P_n\}$ and $\{Q_n\}$ are sequences of operators in $BL(X)$ and $BL(Y)$ respectively such that $\|P_n\|\,\|Q_n\| \le 1$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_k(T'_n) = s_k(T')$ if any of the following holds:*

(i) *X and Y' are separable and $T_n \xrightarrow{wo} T$ as $n \to \infty$.*

(ii) *X is reflexive, Y' is separable and $T'_n \xrightarrow{wo} T'$ as $n \to \infty$.*

4. Approximation under the compactness assumption

In [7], Pietsch discussed the question whether the equality $s_k(T) = s_k(T')$ holds for every $T \in BL(X,Y)$ and $k \in \mathbb{N}$. In [3], Hutton proved that this is true for compact operators T in $BL(X,Y)$ and gave an example to show that this equality need not be true for a non-compact operator. We now make use of the above equality of approximation numbers for compact operators to derive the following results using Corollary 3.6.

Theorem 4.1. *Let $T \in BL(X,Y)$ and $T_n := Q_n T P_n$ for all $n \in \mathbb{N}$, where $\{P_n\}$ and $\{Q_n\}$ be sequences of operators in $BL(X)$ and $BL(Y)$ respectively such that $\|P_n\|\,\|Q_n\| \le 1$ for all $n \in \mathbb{N}$. Suppose X, Y and T_n satisfy any of the conditions* (i) *and* (ii) *of Corollary* 3.6*. Then we have the following:*

(a) *If T_n is compact for every $n \in \mathbb{N}$, then*

$$s_k(T) = s_k(T') \iff \lim_{n\to\infty} s_k(T_n) = s_k(T).$$

(b) *If T is compact, then $\lim_{n\to\infty} s_k(T_n) = s_k(T)$.*

Proof. Part (a) follows from Corollary 3.6 by making use of the fact that $s_k(T_n) = s_k(T_n')$ whenever T_n is compact, and (b) is a consequence of (a) by using the equality $s_k(T) = s_k(T')$ whenever T is compact. □

Theorem 4.2. *Let X, Y, T_n be as in Theorem 4.1 and T be a compact operator from X to Y. If, in addition, $P_n x \to x$ and $Q_n y \to y$ as $n \to \infty$ for all $x \in X$, $y \in Y$ and X and Y satisfy any of the conditions (i) and (ii) of Corollary 3.6, then for each $k \in \mathbb{N}$, $s_k(T_n) \to s_k(T)$ as $n \to \infty$.*

Proof. Since $P_n x \to x$ and $Q_n y \to y$ as $n \to \infty$ for all $x \in X$, $y \in Y$, we obtain that $T_n \to T$ in strong sense. This gives $T_n \xrightarrow{wo} T$ and the result follows from Theorem 4.1 under the condition (i) of Corollary 3.6. When X is reflexive, from Remark 3.5, we have $T_n' \xrightarrow{wo} T'$ whenever $T_n \to T$ in strong sense, and this gives the result under the condition (ii) of Corollary 3.6. □

The following corollary extends the result in [3] so as to include non-compact operators in certain cases.

Corollary 4.3. *Let X, Y, T and T_n be as in Corollary 3.6. Then for each $k \in \mathbb{N}$, $s_k(T) = s_k(T')$ if any of the following holds:*

(a) *X and Y are reflexive and separable and $T_n \to T$ in the strong sense as $n \to \infty$.*
(b) *X and Y' are separable, Y is the dual space of some normed linear space and $T_n \xrightarrow{wo} T$ as $n \to \infty$.*
(c) *X and Y are as in (a) or (b) and $P_n x \to x$ and $Q_n y \to y$ as $n \to \infty$ for all $x \in X$ and $y \in Y$.*

Proof. From Remark 3.5, $T_n \to T$ in the strong sense implies $T_n \xrightarrow{wo} T$ and $T_n' \xrightarrow{wo} T'$ as $n \to \infty$, whenever X is reflexive. Now, under the assumption in (a), the equality $s_k(T) = s_k(T')$ follows from Theorem 4.1 and 2.8, due to the reflexivity and separability of X and Y'. Since $T_n \xrightarrow{wo} T$ implies $T_n \xrightarrow{wo^*} T$ and $T_n' \xrightarrow{wo^*} T'$ as $n \to \infty$, the result follows from Theorem 3.3 and 4.1, under the assumption in (b). The assumptions in (c) gives the strong convergence of T_n to T. Hence the result follows from (a) and (b) in this particular case. □

In [3], Hutton proved the following proposition to establish that $s_k(T)$ need not be equal to $s_k(T')$ for a general non-compact operator T.

Proposition 4.4. (cf. [3], Proposition 2.3) *If $I_E : \ell^1 \to c_0$ and $I_F : \ell^1 \to \ell^\infty$ are the natural injections, then $s_k(I_E) = 1$ for each $k \in \mathbb{N}$ and $s_k(I_F) = 1/2$ for each $k \in \{2, 3, \ldots\}$.*

Note that the transpose of the natural injection $I_E : \ell^1 \to c_0$ is linearly isometric with $I_F : \ell^1 \to \ell^\infty$. Now, we make use of Proposition 4.4 to establish that $\{s_k(Q_n T P_n)\}$ need not converge to $s_k(T)$ for a non-compact operator $T \in BL(X, Y)$ if the codomain is not the dual space of some separable space. To see this, let $T = I_E$ and let $P_n \in BL(\ell^1)$ and $Q_n \in BL(c_0)$ be the projection operators defined by $(x(1), x(2), \ldots) \to (x(1), x(2), \ldots, x(n), 0, 0, \ldots)$ on ℓ^1 and c_0, respectively. Then $T_n := Q_n T P_n$ satisfy all the assumptions in Corollary 3.6(i). By Proposition 4.4, $s_k(T) \neq s_k(T')$. Hence, Theorem 4.1(a) shows that $s_k(Q_n T P_n) \not\to s_k(T)$ as $n \to \infty$.

Remark 4.5. From Corollary 4.3(b) and Proposition 4.4, it follows that c_0 is a not linearly isometric with the dual of any normed linear space.

Remark 4.6. We would like to remark that there are spaces X and Y admitting the sequences (P_n) and (Q_n) of operators in $BL(X)$ and $BL(Y)$, respectively, satisfying the conditions of Corollary 4.3. For example, if $Y = \ell^p$ with $1 < p < \infty$, which is a reflexive space, then for any separable normed linear space X and for any $T \in BL(X, Y)$, we have $s_k(T) = s_k(T')$, for in this case we may take $P_n = I$ for all $n \in \mathbb{N}$ and

$$Q_n x = (x(1), x(2), \ldots, x(n), 0, 0, \ldots), \quad x = (x(1), x(2), \ldots) \in \ell^p.$$

Remark 4.7. Since the strict inclusion map from X to Y is used in Proposition 4.4 for getting the counter examples, it is of interest to see if the conclusions in Theorem 3.3 and 4.1 hold if X and Y are the same normed linear spaces, by removing the additional assumptions on the codomain. This still remains an open question.

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