J. Math. Anal. Appl. 464 (2018) 939–954

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The (n, ϵ)-pseudospectrum of an element of a Banach algebra

Kousik Dhara *, S.H. Kulkarni

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

A R T I C L E I N F O

Article history:
Received 22 December 2017
Available online 19 April 2018
Submitted by T. Ransford

Keywords:
Banach algebra
Spectrum
Pseudospectrum
(n, ϵ)-pseudospectrum

A B S T R A C T

Let A be a complex unital Banach algebra, $a \in A$, $n \in \mathbb{Z}_+$ and $\epsilon > 0$. The (n, ϵ)-pseudospectrum $\Lambda_{n,\epsilon}(a)$ of a is defined as

$$\Lambda_{n,\epsilon}(a) := \sigma(a) \cup \{\lambda \notin \sigma(a) : \|(\lambda - a)^{-2^n}\|^{1/2^n} \geq \tfrac{1}{\epsilon}\}.$$

Here $\sigma(a)$ denotes the spectrum of a. The usual pseudospectrum $\Lambda_\epsilon(a)$ of a is a special case of this, namely $\Lambda_{0,\epsilon}(a)$. It is proved that (n, ϵ)-pseudospectrum approximates the closed ϵ-neighbourhood of spectrum for large n. Further, it has been shown that (n, ϵ)-pseudospectrum has no isolated points, has a finite number of connected components and each component contains an element from $\sigma(a)$. Some examples are given to illustrate these results.

1. Introduction

Let us begin with the following motivating example, first considered by E.B. Davies (see [7]):

for $\delta \in \mathbb{R}$, let $A_\delta : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be defined by

$$(A_\delta x)(n) = \begin{cases} \delta x(n+1), & n = 0 \\ x(n+1), & n \neq 0, \ n \in \mathbb{Z}. \end{cases} \tag{1}$$

For $\delta \neq 0$, the spectrum $\sigma(A_\delta) = \{z \in \mathbb{C} : |z| = 1\}$ and for $\delta = 0$, $\sigma(A_0) = \{z \in \mathbb{C} : |z| \leq 1\}$. Observe that $A_\delta \to A_0$ as $\delta \to 0$, but the Hausdorff distance $d_H(\sigma(A_\delta), \sigma(A_0)) = 1$ for all $\delta \neq 0$.

As Hansen observed in [10], this situation is of concern to a numerical analyst because if one attempts to compute the spectrum of an operator using numerical methods on a computer, then due to round off and

* Corresponding author.
E-mail addresses: kousik.dhara1@gmail.com (K. Dhara), shk@iitm.ac.in (S.H. Kulkarni).

https://doi.org/10.1016/j.jmaa.2018.04.043
0022-247X/

other errors, the result that one would get will be the spectrum of a slightly perturbed operator. But as the above example shows, this may be quite away from the spectrum of the original operator. In order to avoid this kind of discontinuous behaviour of the spectrum and to find a better approximation of spectrum, Hansen first introduced the concept of (n, ϵ)-pseudospectrum in [10] for operators on separable Hilbert spaces and investigated further its approximating properties in [11] and [12]. The theory of (n, ϵ)-pseudospectrum extends the well known theory of ϵ-pseudospectrum. This was introduced for studying non-normal matrices.

A detailed treatment of ϵ-pseudospectrum for matrices and operators along with several applications can be found in [22] and [23]. Many authors have studied ϵ-pseudospectra of operators on Hilbert spaces and Banach spaces (for instance, see [1], [4], [5], [6], [15] and [16]). Recently ϵ-pseudospectrum of an element of an arbitrary Banach algebra has been studied elaborately in [14].

Hansen originally described the (n, ϵ)-pseudospectra for operators using the involution on operators on Hilbert spaces and computed (n, ϵ)-pseudospectra of some finite matrices using singular values. Since the operators on a Banach space do not have involution, M. Seidel presented a different theoretical approach using rectangular finite sections in [19]. His approach made use of the concepts of injection and surjection modulus.

As the theory of (n, ϵ)-pseudospectra progresses, it is important to study this notion in a more general setting. In a recent pre-print, A.C. Hansen and O. Nevanlinna (see [13]) have defined and extended the notion of (n, ϵ)-pseudospectra to an arbitrary Banach algebra and mentioned the complexities in extending the main approximating theorem i.e. Theorem 2.1. In this note, we aim to study this concept in a Banach algebra in a systematic way by proving several important results.

In Section 2, some elementary properties of (n, ϵ)-pseudospectrum of an element of a Banach algebra are discussed. Whereas some of these (such as Theorem 2.8, Corollary 2.23) are analogous to the properties of pseudospectrum, in some cases (for example, Theorem 2.13) a few additional conditions are added to get the desired result for the general case. Also, it is shown that (n, ϵ)-pseudospectrum approximates the closed ϵ-neighbourhood of $\sigma(a)$ as n grows larger (Theorem 2.8 (3)). This provides an important tool for spectral approximation (Remark 2.9). G_n-class elements are introduced. Using functional calculus for derivatives, the scalar elements are also characterized in terms of their (n, ϵ)-pseudospectra (Corollary 2.23).

A few topological aspects of (n, ϵ)-pseudospectrum are discussed in Section 3. For a fixed element a in a Banach algebra A and $n \in \mathbb{Z}_+$, the map $\epsilon \mapsto \Lambda_{n,\epsilon}(a)$ is right continuous (Theorem 3.1). Also it is shown that $\Lambda_{n,\epsilon}(a)$ has no isolated points, has a finite number of components and each component contains an element from $\sigma(a)$ (Theorem 3.3 and Theorem 3.4). Finally, as an application and illustration, we have considered the operator A_δ in Example (1) and established that $d_H(\Lambda_{n,\epsilon}(A_\delta), \sigma(A_\delta) + D(0, \epsilon)) \to 0$ as $n \to \infty$. Also, it is shown explicitly that $d_H(\Lambda_{n,\epsilon}(A_\delta), \Lambda_{n,\epsilon}(A_0)) \to 0$ as $\delta \to 0$. The question whether this happens also in case of an arbitrary Banach algebra, or in other words whether the map $a \mapsto \Lambda_{n,\epsilon}(a)$ is continuous needs further detailed investigation. This will be taken up in future.

2. Definitions and some elementary properties of (n, ϵ)-pseudospectrum

Notation: Throughout the paper we will use the following notation:

$$B(\mu, r) := \{z \in \mathbb{C} : |z - \mu| < r\},\ D(\mu, r) := \{z \in \mathbb{C} : |z - \mu| \le r\}. \text{ For } \Omega \subseteq \mathbb{C},\ \Omega + D(0, r) := \bigcup_{\mu \in \Omega} D(\mu, r).$$

For $\Omega \subseteq \mathbb{C}$, $\overline{\Omega}$ and $\delta\Omega$ denote the closure and the boundary of Ω respectively.

$B(H)$ denotes the set of all bounded linear operators on a separable Hilbert space H. $K(\mathbb{C})$ denotes the set of all non-empty compact subsets of $\mathbb{C}$. For $K_1, K_2 \in K(\mathbb{C})$, the Hausdorff distance between K_1, K_2 is defined by

$$d_H(K_1, K_2) = \max\{\sup_{a \in K_1} d(a, K_2), \sup_{b \in K_2} d(b, K_1)\}$$

where $d(a,K) := \inf_{k\in K}|a-k|$. Then $(K(\mathbb{C}), d_H)$ is a metric space.
$\mathbb{C}^{n\times n}$ denotes the space of $n\times n$ matrices over $\mathbb{C}$.

Further, unless specified otherwise, A will always denote a complex Banach algebra with identity 1. We identify λ with $\lambda.1$. Let $a\in A$. The spectrum of a is denoted by $\sigma(a)$ and is defined as

$$\sigma(a) := \{\lambda\in\mathbb{C} : \lambda - a \text{ is not invertible}\}.$$

The spectral radius a is denoted by $r(a)$ and is defined as

$$r(a) := \sup\{|\lambda| : \lambda\in\sigma(a)\}.$$

A Banach algebra A with an involution $a\to a^*$ which satisfies

$$\|a^*a\| = \|a\|^2 \quad \forall a\in A$$

is called a C^*-algebra.
For $\epsilon>0$, the ϵ-pseudospectrum of a is denoted by $\Lambda_\epsilon(a)$ and is defined as

$$\Lambda_\epsilon(a) := \sigma(a)\cup\{\lambda\notin\sigma(a) : \|(\lambda-a)^{-1}\| \geq \frac{1}{\epsilon}\}.$$

Hansen (2008) defined the (n,ϵ)-pseudospectrum of $T\in B(H)$ as the set

$$\Lambda^*_{n,\epsilon}(T) := \sigma(T)\cup\{\lambda\notin\sigma(T) : \|(\lambda-T)^{-2^n}\|^{1/2^n} > \frac{1}{\epsilon}\}.$$

First we shall mention the following theorem of Hansen which provides some basic properties of the (n,ϵ)-pseudospectrum of a bounded linear operator on a Hilbert space. To start with, let $T\in B(H)$.For $n\in\mathbb{Z}_+$, define $\gamma_n(T,.) : \mathbb{C}\to[0,\infty]$ by

$$\gamma_n(T,z) = \min[\min\{\lambda^{1/2^{n+1}} : \lambda\in\sigma(S(z)^*S(z))\},\ \min\{\lambda^{1/2^{n+1}} : \lambda\in\sigma(S(z)S(z)^*)\}], \tag{2}$$

where $S(z) = (T-z)^{2^n}$.

Theorem 2.1 *(Hansen). Let $T\in B(H)$ and $\epsilon>0$. Let $n\in\mathbb{Z}_+$ and $\gamma_n(T,.)$ be as in (2). Then the following holds:*

1. $\Lambda^*_{n+1,\epsilon}(T)\subseteq\Lambda^*_{n,\epsilon}(T)$,
2. $\overline{\Lambda^*_{n,\epsilon}(T) = \{z\in\mathbb{C} : \gamma_n(T,z)<\epsilon\}}$,
3. $\overline{\{z\in\mathbb{C} : \gamma_n(T,z)<\epsilon\}} = \{z\in\mathbb{C} : \gamma_n(T,z)\leq\epsilon\}$,
4. $d_H(\overline{\Lambda^*_{n,\epsilon}(T)}, \overline{\sigma(T)+B(0,\epsilon)})\to 0$ *when* $n\to\infty$, *and*
5. *if* $\{T_k\}\subseteq B(H)$ *is such that* $T_k\to T$ *in norm as* $k\to\infty$, *then* $d_H(\overline{\Lambda^*_{n,\epsilon}(T_k)}, \overline{\Lambda^*_{n,\epsilon}(T)})\to 0$ *when* $k\to\infty$.

The properties 2 and 3 are based on the functions γ_n which in turn depend on involution. While studying these properties in a Banach algebra, Hansen and Nevanlinna mentioned a few difficulties in [13]. In this note, we shall present these properties in modified forms with elementary proofs.

Definition 2.2. Let A be a Banach algebra and $a\in A$. For $n\in\mathbb{Z}_+\cup\{0\}$ and $z\in\mathbb{C}$, define functions $\gamma_n(a,z)$ and $\gamma(a,z)$ by

$$\gamma_n(a,z) = \begin{cases} \|(z-a)^{-2^n}\|^{-1/2^n} & \text{, if } z \notin \sigma(a) \\ 0 & \text{, if } z \in \sigma(a) \end{cases}$$

and $\gamma(a,z) = d(z, \sigma(a))$.

The next theorem says that the newly defined functions γ_n coincide with Hansen's definition as in (2) when the underlying space is a C^*-algebra.

Theorem 2.3. *Let A be a C^*-algebra, $a \in A$ and $n \in \mathbb{Z}_+$. Then $\forall z \in \mathbb{C}$,*

$$\gamma_n(a,z) = \min[\min\{\lambda^{1/2^{n+1}} : \lambda \in \sigma(S(z)^*S(z))\},\ \min\{\lambda^{1/2^{n+1}} : \lambda \in \sigma(S(z)S(z)^*)\}]$$

where $S(z) = (a-z)^{2^n}$.

Proof. For the sake of simplicity, let us denote the right hand side of the expression by $\delta_n(a,z)$. Let us consider the following two cases.

Case 1: Suppose $z \notin \sigma(a)$. Let $S := (a-z)^{2^n}$. Then S is invertible and so are S^*S and SS^*. Thus $0 \notin \sigma(S^*S)$ and $0 \notin \sigma(SS^*)$. Again, we know that the non zero spectral values of S^*S and SS^* are same and their spectrum lie in the positive real line. Suppose $\delta_n(a,z) = \lambda_0^{1/2^{n+1}}$. Note that $\sigma((S^*S)^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma(S^*S)\}$. Thus we have,

$$\|(S^*S)^{-1}\| = r((S^*S)^{-1}) = \max\{\frac{1}{\lambda} : \lambda \in \sigma(S^*S)\} = \frac{1}{\min\{\lambda : \lambda \in \sigma(S^*S)\}} = \frac{1}{\lambda_0}.$$

Finally we get,

$$\begin{aligned} \delta_n(a,z) &= \lambda_0^{1/2^{n+1}} \\ &= \frac{1}{\|(S^*S)^{-1}\|^{1/2^{n+1}}} \\ &= \frac{1}{\|S^{-1}\|^{1/2^n}} \\ &= \frac{1}{\|(a-z)^{-2^n}\|^{1/2^n}}. \end{aligned}$$

Case 2: Suppose $z \in \sigma(a)$. Then $S := (a-z)^{2^n}$ is not invertible. Hence S^* is not invertible. This means either SS^* or S^*S is not invertible, for otherwise S^* would have both left and right inverse and hence invertible. So either $0 \in \sigma(SS^*)$ or $0 \in \sigma(S^*S)$. Hence $\delta_n(a,z) = 0$. □

Remark 2.4. In [12], Hansen proved the second part of the proof of the preceding theorem for $T \in B(H)$ using polar decomposition and considered various cases. However, our proof is elementary and applicable to any C^*-algebras.

Since we shall make use of properties of the functions $\gamma_n(a,z)$ and $\gamma(a,z)$ in many proofs, we collect some important properties in the following Proposition.

Proposition 2.5. *Let A be a unital Banach algebra, $a \in A$ and $n \in \mathbb{Z}_+ \cup \{0\}$. Then the following holds:*

1. $\gamma_n(\lambda, z) = |\lambda - z|\ \forall \lambda, z \in \mathbb{C}$.
2. $\gamma_n(a,z)$ *and* $\gamma(a,z)$ *are continuous* $\forall z \in \mathbb{C}$.

3. $\gamma_n(a,z) \le \gamma_{n+1}(a,z) \le \gamma(a,z)\ \forall z \in \mathbb{C}$.
4. $\gamma_n(a,z)$ *converges to* $\gamma(a,z)\ \forall z \in \mathbb{C}$. *The convergence is uniform on compact subsets of* $\mathbb{C}$.
5. $\gamma_n(a+\lambda,z) = \gamma_n(a,z-\lambda)\ \forall \lambda \in \mathbb{C}$.
6. $\gamma_n(\lambda a,z) = \frac{1}{|\lambda|}\gamma_n(a,\frac{z}{\lambda})\ \forall z \in \mathbb{C}$ *and* $\lambda \in \mathbb{C}\setminus\{0\}$.
7. $|z| - \|a\| \le \gamma_0(a,z)\ \forall z \in \mathbb{C}$. *If* a *is invertible, then* $\frac{1}{\|a^{-1}\|} - |z| \le \gamma_0(a,z)\ \forall z \in \mathbb{C}$.

Proof. The proofs are simple. We indicate the proofs for the sake of completeness.

1. Straightforward from definition.
2. Clearly, $\gamma(a,z)$ is continuous $\forall z \in \mathbb{C}$.
   Let $z_0 \in \delta\sigma(a)$. Suppose $\{z_m\} \subseteq \mathbb{C}\setminus\sigma(a)$ is a sequence such that $z_m \to z_0$. Then $\gamma_n(z_m) = \|(z_m - a)^{-2^n}\|^{-1/2^n} \to 0 = \gamma(z_0)$ as $m \to \infty$, proving continuity at z_0. The continuity at all other points in $\mathbb{C}$ is obvious.
3. For any $b \in A$, observe that $\{\|b^{2^m}\|^{1/2^m}\}$ is a decreasing sequence and by spectral radius formula, we have $r(b) = \lim_{m\to\infty}\|b^{2^m}\|^{1/2^m}$. So, $r(b) \le \|b^{2^m}\|^{1/2^m}\ \forall m \in \mathbb{Z}_+$. For $z \notin \sigma(a)$,
   $$d(z,\sigma(a)) = \frac{1}{r((z-a)^{-1})} \ge \frac{1}{\|(z-a)^{-2^m}\|^{1/2^m}}\ \forall m \in \mathbb{Z}_+ \text{ [Replace } b \text{ by } (z-a)^{-1}].$$
   Hence, $\gamma_n(a,z) \le \gamma_{n+1}(a,z) \le \gamma(a,z)\ \forall z \in \mathbb{C}\setminus\sigma(a)$. For $z \in \sigma(a)$, $\gamma_n(a,z) = 0 = \gamma_{n+1}(a,z) = \gamma(a,z)$.
4. First note that for $z \in \sigma(a)$, $\gamma_n(a,z) = 0 = \gamma(a,z)\ \forall n$. Next, let $z \notin \sigma(a)$ and $b = (z-a)^{-1}$. Then
   $$\lim_{n\to\infty}\gamma_n(a,z) = \lim_{n\to\infty}\frac{1}{\|b^{2^n}\|^{1/2^n}} = \frac{1}{r(b)} = \frac{1}{r((a-z)^{-1})} = d(z,\sigma(a)) = \gamma(a,z).$$
   Thus $\gamma_n(a,z) \to \gamma(a,z)\ \forall z \in \mathbb{C}$. Applying Dini's theorem, $\gamma_n(a,.)$ converges to $\gamma(a,.)$ uniformly on compact subsets of $\mathbb{C}$.
5. By spectral mapping theorem, we have $\sigma(a+\lambda) = \lambda + \sigma(a)\ \forall \lambda \in \mathbb{C}$. Thus, $\forall \lambda, z \in \mathbb{C}$, we have
   $$\begin{aligned}\gamma_n(a+\lambda,z) &= \begin{cases}\|\{(z-\lambda)-a\}^{-2^n}\|^{-1/2^n} & \text{, if } z-\lambda \notin \sigma(a)\\ 0 & \text{, if } z-\lambda \in \sigma(a)\end{cases}\\ &= \gamma_n(a,z-\lambda).\end{aligned}$$
6. Note that $z \in \sigma(\lambda a) \iff \frac{z}{\lambda} \in \sigma(a)$ for $\lambda \in \mathbb{C}\setminus\{0\}$. So, $\forall z \in \mathbb{C}$ and $\lambda \in \mathbb{C}\setminus\{0\}$, we have
   $$\begin{aligned}\gamma_n(\lambda a,z) &= \begin{cases}|\lambda|\|(\frac{z}{\lambda}-a)^{-2^n}\|^{-1/2^n} & \text{, if } \frac{z}{\lambda} \notin \sigma(a)\\ 0 & \text{, if } \frac{z}{\lambda} \in \sigma(a)\end{cases}\\ &= |\lambda|\gamma_n(a,\frac{z}{\lambda}).\end{aligned}$$
7. Let $|z| > \|a\|$. Then $\|(z-a)^{-1}\| \le \frac{1}{|z|-\|a\|}$ and the first part follows. For the second part, let $|z| < \frac{1}{\|a^{-1}\|}$. Then $(1 - za^{-1})$ is invertible and
   $$\begin{aligned}\|(z-a)^{-1}\| &= \|a^{-1}(1-za^{-1})^{-1}\|\\ &\le \frac{\|a^{-1}\|}{1-|z|\|a^{-1}\|}.\end{aligned}$$
   So, $\frac{1}{\|a^{-1}\|} - |z| \le \|(z-a^{-1})\|^{-1}$. Consequently, 7 follows. □

Definition 2.6. Let A be a unital Banach algebra and $a \in A$. For $\epsilon > 0$ and $n \in \mathbb{Z}_+$, the (n,ϵ)-pseudospectrum of a is defined by

$$\Lambda_{n,\epsilon}(a) := \sigma(a) \cup \{\lambda \notin \sigma(a) : \|(\lambda - a)^{-2^n}\|^{1/2^n} \geq \frac{1}{\epsilon}\}.$$

Equivalently, $\Lambda_{n,\epsilon}(a) := \{\lambda \in \mathbb{C} : \gamma_n(a,\lambda) \leq \epsilon\}$.

Remark 2.7.

1. We observe that the $(0,\epsilon)$-pseudospectrum is nothing but the usual ϵ-pseudospectrum.
2. We have used '$\geq$' sign in the definition of (n,ϵ)-pseudospectrum instead of '$>$' sign. Many authors (see [14]) use the former definition for the case $n = 0$. Also '$\geq$' sign has been used in [19] to define the (n,ϵ)-pseudospectrum for operators on complex Banach spaces. A discussion regarding these two definitions (for $n = 0$) can be found in [14] and [21].

The following theorem provides some elementary properties of the (n,ϵ)-pseudospectrum.

Theorem 2.8. *Let A be a Banach algebra. Let $a \in A$, $n \in \mathbb{Z}_+$ and $\epsilon > 0$. Then the following results hold:*

1. $\Lambda_{n,\epsilon}(\lambda) = D(\lambda,\epsilon)\ \forall \lambda \in \mathbb{C}$.
2. $\Lambda_{n+1,\epsilon}(a) \subseteq \Lambda_{n,\epsilon}(a)$.
3. $\sigma(a) + D(0,\epsilon) = \bigcap_{n\in\mathbb{Z}_+} \Lambda_{n,\epsilon}(a)$. *Further,* $d_H(\Lambda_{n,\epsilon}(a), \sigma(a) + D(0,\epsilon)) \to 0$ *as* $n \to \infty$.
4. $\sigma(a) = \bigcap_{\epsilon>0} \Lambda_{n,\epsilon}(a)$.
5. $\Lambda_{n,\epsilon_1}(a) \subseteq \Lambda_{n,\epsilon_2}(a)$ *for* $0 < \epsilon_1 < \epsilon_2$.
6. $\Lambda_{n,\epsilon}(a + \lambda) = \lambda + \Lambda_{n,\epsilon}(a)$ *for* $\lambda \in \mathbb{C}$.
7. $\Lambda_{n,\epsilon}(\lambda a) = \lambda\Lambda_{n,\frac{\epsilon}{|\lambda|}}(a)$ *for* $\lambda \in \mathbb{C} \setminus \{0\}$.
8. $\Lambda_{n,\epsilon}(a) \subseteq D(0, \|a\| + \epsilon)$. *Further, if a is invertible and* $0 < \epsilon < \frac{1}{\|a^{-1}\|}$, *then*
$$\Lambda_{n,\epsilon}(a) \subseteq \{z \in \mathbb{C} : \frac{1}{\|a^{-1}\|} - \epsilon \leq |z| \leq \|a\| + \epsilon\}.$$
9. $\Lambda_{n,\epsilon}(a)$ *is a non-empty compact subset of* $\mathbb{C}$.

Proof. Proofs of 1, 2, 6, 7 and 8 directly follow from 1, 3, 5, 6 and 7 respectively of Proposition 2.5. Also 5 follows directly.

To prove 3, we use 3 and 4 of Proposition 2.5. Note that

$$\begin{aligned} \lambda \in \sigma(a) + D(0,\epsilon) &\iff \gamma(a,\lambda) \leq \epsilon \\ &\iff \gamma_n(a,\lambda) \leq \epsilon\ \forall n \in \mathbb{Z}_+ \\ &\iff \lambda \in \Lambda_{n,\epsilon}(a)\ \forall n \in \mathbb{Z}_+. \end{aligned}$$

Hence the equality follows. The latter part follows from the fact that a decreasing sequence of non-empty compact sets converges to their intersection in Hausdorff metric.

To prove 4, we observe that

$$\begin{aligned} \lambda \in \bigcap_{\epsilon>0} \Lambda_{n,\epsilon}(a) &\iff \gamma_n(\lambda,a) \leq \epsilon\ \forall \epsilon > 0 \\ &\iff \gamma_n(\lambda,a) = 0 \\ &\iff \lambda \in \sigma(a). \end{aligned}$$

For proving 9, we note that $\Lambda_{n,\epsilon}(a)$ is closed as $\gamma_n(a,.)$ is continuous (by 4 of Proposition 2.5), bounded by 8 and non empty as it contains $\sigma(a)$. $\square$

Remark 2.9. The result 3 of Theorem 2.8 is about the approximation of the closed ϵ-neighbourhood $\sigma(a) + D(0,\epsilon)$ of the spectrum. It says that if we have a good method of computing $\Lambda_{n,\epsilon}(a)$, then we can get information about $\sigma(a)$. This aspect of computing (n,ϵ)-pseudospectrum is discussed in [11] for bounded operators on separable Hilbert spaces. This involves the use of the functions $\gamma_n(T,.)$. A version of this result is known in the literature for operators on Hilbert spaces and Banach spaces. It was also observed in [19], that this can be extended to the general case of elements of a Banach algebra. Our proof is more elementary.

Another way of approximation is given in Theorem 2 in [19] for operators on Banach spaces. The next theorem shows that a similar result is true for elements of a Banach algebra. Its proof is also similar to the one given in [19].

Theorem 2.10. *Let A be a unital Banach algebra and $a \in A$. Then for $\eta > \epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$,*

$$\sigma(a) + D(0,\epsilon) \subseteq \Lambda_{n,\epsilon}(a) \subseteq \sigma(a) + D(0,\eta).$$

Proof. Let $\eta > \epsilon > 0$. Note that the first inclusion follows from 3 of Theorem 2.8 for all n. To prove the second inclusion, let $R > \|a\| + \eta$. Then $\sigma(a) + D(0,\eta) \subseteq D(0,R)$. Applying Proposition 2.5, we have uniform convergence of γ_n to γ on $D(0,R)$. Thus, $\exists n_0 \in \mathbb{N}$ such that

$$\gamma(a,z) - (\eta - \epsilon) < \gamma_n(a,z) \text{ for all } z \in D(0,R) \text{ and } n \geq n_0.$$

Let $z \in \Lambda_{n,\epsilon}(a)$. Then, for $n \geq n_0$, $\gamma(a,z) \leq \gamma_n(a,z) + (\eta - \epsilon) \leq \eta$, i.e., $d(z,\sigma(a)) \leq \eta$. Hence $z \in \sigma(a) + D(0,\eta)$. $\square$

Remark 2.11. We note that in general, $\Lambda_{n,\epsilon}(a) \neq \overline{\Lambda^*_{n,\epsilon}(a)}$. In other words, $\|(\lambda - a)^{-2^n}\|$ can be constant on open sets. There are known examples of this given in [2] and [20]. Also it follows from Theorem 2.10 that the level set $\{\lambda \in \mathbb{C} : \gamma_n(a,\lambda) = \epsilon\}$ becomes small for large n. This was shown by Seidel in [19] for a bounded linear operator on a Banach space X.

Remark 2.12. The following property is known for ϵ-pseudospectra (see [8]):

$$\bigcup_{\|b\| \leq \epsilon} \sigma(a+b) \subseteq \Lambda_\epsilon(a). \tag{3}$$

Moreover, under certain conditions, this inclusion becomes an equality and thus offers a method to approximate the pseudospectrum of a by considering spectra of small perturbations of a. This is not true for (n,ϵ)-pseudospectra in general. See Example 2.14 below. We consider a much weaker version of the above property for (n,ϵ)-pseudospectra in the following:

Proposition 2.13. *Suppose A is a Banach algebra, $a \in A$ and $\epsilon > 0$. Suppose $b \in A$ is such that $ab = ba$ and $\|b^{2^n}\|^{1/2^n} \leq \epsilon$ for some $n \in \mathbb{Z}_+$. Then*

$$\sigma(a+b) \subseteq \sigma(a) + D(0,\epsilon) \subseteq \Lambda_{m,\epsilon}(a)\ \forall m.$$

Proof. Let $\mu \in \sigma(a+b)$. Since $ab = ba$, $\sigma(a+b) \subseteq \sigma(a) + \sigma(b)$ (see Theorem 11.23 of [18]). Thus $\mu = z + w$ for some $z \in \sigma(a)$ and $w \in \sigma(b)$. Hence $|w| \leq r(b) \leq \|b^{2^n}\|^{1/2^n} \leq \epsilon$. The other part follows from 3 of Theorem 2.8. $\square$

The commutativity of a and b can not be dropped from the preceding theorem as the following example shows.

Example 2.14. Let $A = \mathbb{C}^{2\times 2}$ with $\|.\|_1$ norm (maximum absolute column sum). Consider $a = \begin{pmatrix} 0 & 9 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $a^2 = b^2 = 0$ and $ab \neq ba$. Also $\sigma(a+b) = \{3, -3\}$. Observe that $\|\alpha a + \beta\|_1 = 9|\alpha| + |\beta|$ $\forall \alpha, \beta \in \mathbb{C}$. Let $\lambda \notin \sigma(a) = \{0\}$. We have $(\lambda - a)^{-1} = \frac{1}{\lambda} + \frac{a}{\lambda^2}$. Let $m \in \mathbb{N}$. Then

$$(\lambda - a)^{-m} = (\frac{1}{\lambda} + \frac{a}{\lambda^2})^m = \frac{1}{\lambda^m} + \frac{ma}{\lambda^{m+1}}$$

and consequently $\|(\lambda - a)^{-m}\|_1 = \frac{1}{|\lambda|^m} + \frac{9m}{|\lambda|^{m+1}}$. Take $m = 2^n$. Then

$$\begin{aligned} \lambda \in \Lambda_{n,\epsilon}(a) &\iff \frac{|\lambda| + 9 \times 2^n}{|\lambda|^{2^n+1}} \geq \frac{1}{\epsilon^{2^n}} \\ &\iff |\lambda|^{2^n+1} \leq (|\lambda| + 9 \times 2^n)\epsilon^{2^n}. \end{aligned}$$

Choose $\epsilon = 1$ and $n = 1$. Then it follows that $3 \notin \Lambda_{1,1}(a)$.

Remark 2.15. The inclusion $\sigma(a) + D(0, \epsilon) \subseteq \Lambda_{n,\epsilon}(a)$ can be proper. We give an example below.

Example 2.16. Let $\mathcal{A} = \{a \in \mathbb{C}^{2\times 2} : a = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\}$ with norm given by $\|a\| = |x| + |y|$. It is clear that $\mathcal{A}$ is a Banach algebra. Let us consider the element $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Observe that $a^2 = 0$ and $\|\alpha a + \beta\| = |\alpha| + |\beta|$ $\forall \alpha, \beta \in \mathbb{C}$.

Let $\lambda \notin \sigma(a) = \{0\}$. Then $(\lambda - a)^{-1} = \frac{1}{\lambda} + \frac{a}{\lambda^2}$ and so $\|(\lambda - a)^{-1}\| = \frac{1}{|\lambda|^2}\|\lambda + a\| = \frac{1+|\lambda|}{|\lambda|^2}$.

Note that

$$\begin{aligned} \lambda \in \Lambda_\epsilon(a) &\iff |\lambda|^2 \leq \epsilon + |\lambda|\epsilon \\ &\iff (|\lambda| - \frac{\epsilon}{2})^2 \leq \epsilon + \frac{\epsilon^2}{4} \\ &\iff |\lambda| \leq \frac{\epsilon}{2} + \sqrt{\epsilon + \frac{\epsilon^2}{4}}. \end{aligned}$$

Thus, $\Lambda_\epsilon(a) = D(0, \frac{\epsilon}{2} + \sqrt{\epsilon + \frac{\epsilon^2}{4}}) \supsetneqq D(0, \epsilon) = \sigma(a) + D(0, \epsilon)$. For $n \in \mathbb{Z}_+$, it can be verified (similar to Example 2.14) that

$$\lambda \in \Lambda_{n,\epsilon}(a) \iff |\lambda| \leq (|\lambda| + 2^n)^{\frac{1}{2^n+1}} \epsilon^{\frac{2^n}{2^n+1}}.$$

Thus, if λ, ϵ and n are such that $\epsilon < |\lambda| \leq (|\lambda| + 2^n)^{\frac{1}{2^n+1}} \epsilon^{\frac{2^n}{2^n+1}}$, then $\lambda \in \Lambda_{n,\epsilon}(a)$ but $\lambda \notin \sigma(a) + D(0, \epsilon)$. For example, suppose $n = 1, \epsilon = \frac{1}{2}$ and consider λ such that $\frac{1}{2} < |\lambda| < \frac{2}{3}$. Then $\lambda \in \Lambda_{n,\epsilon}(a)$ but $\lambda \notin \sigma(a) + D(0, \epsilon)$.

Again $\lambda \in \Lambda_{n,\epsilon}(a) \subseteq D(0, 1 + \epsilon) \Rightarrow |\lambda| \leq 1 + \epsilon$.

So $\lambda \in \Lambda_{n,\epsilon}(a) \Rightarrow |\lambda| \leq (1 + \epsilon + 2^n)^{\frac{1}{2^n+1}} \epsilon^{\frac{2^n}{2^n+1}}$.

Choose k such that $n \geq k \Rightarrow (1 + \epsilon + 2^n) \leq 2^{n+1}$.

Then for $n \geq k$, $\Lambda_{n,\epsilon}(a) \subseteq D(0, R_n)$ where $R_n = 2^{\frac{n+1}{2^n+1}} \epsilon^{\frac{2^n}{2^n+1}}$. Note that $R_n \to \epsilon$ as $n \to \infty$.

Thus, for $n \geq k$, $d_H(\sigma(a) + D(0, \epsilon), \Lambda_{n,\epsilon}(a)) \leq d_H(D(0, \epsilon), D(0, R_n)) = R_n - \epsilon \to 0$ as $n \to \infty$.

This illustrates the abstract result 3 of Theorem 2.8.

Definition 2.17. Let A be a unital Banach algebra and $n \in \mathbb{Z}_+$. An element $a \in A$ is said to be of G_n-class if $\gamma_{n-1}(a, \lambda) = d(\lambda, \sigma(a))$ $\forall \lambda \in \mathbb{C}$.

Remark 2.18. It is immediate from the above definition that a is of G_n-class iff $\Lambda_{n-1,\epsilon}(a) = \sigma(a) + D(0,\epsilon)\ \forall\epsilon > 0$. For $n = 1$, the above definition coincides with the familiar definition of G_1-class (see [14] and [17]). It is known that every normal element in a C^*-algebra is of G_1-class.

Remark 2.19. By Proposition 2.5, we recall that

$$\gamma_{n-1}(a,z) \le \gamma_n(a,z) \le d(z,\sigma(a))\ \forall z \in \mathbb{C}.$$

Hence, G_n-class elements are contained in G_{n+1}-class elements.

Remark 2.20. In algebra $\mathcal{A}$ of Example 2.16, $b = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \in G_n$ iff $y = 0$. Indeed,

$$\begin{aligned} b \in G_n &\iff \|(\lambda - b)^{-2^{n-1}}\|^{1/2^{n-1}} = \frac{1}{|\lambda - x|}\ \forall\lambda \ne x \\ &\iff \|(1 - \frac{ya}{(\lambda - x)})^{-2^{n-1}}\|^{1/2^{n-1}} = 1\ \forall\lambda \ne x \\ &\iff \|1 + \frac{ya}{\lambda - x}2^{n-1}\| = 1\ \forall\lambda \ne x \\ &\iff 1 + 2^{n-1}|\frac{y}{\lambda - x}| = 1\ \forall\lambda \ne x \\ &\iff y = 0. \end{aligned}$$

Thus, in this algebra, G_1 and G_n-class elements are same $\forall n$.

However, the inclusion $G_n \subseteq G_{n+1}$ can be proper for some n. Let us consider the following example.

Example 2.21. Consider the matrix $a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in the algebra $\mathcal{A}$ of Example 2.16. Note that a is not of G_n-class for any n. Recall that $\gamma_0(a,\lambda) = \frac{|\lambda|^2}{1+|\lambda|} \ge \frac{|\lambda|^2-1}{|\lambda|+1} = |\lambda| - 1$ for $|\lambda| \ge 1$. Thus

$$|\lambda| - 1 \le \gamma_0(a,\lambda) \le \gamma_n(a,\lambda) \le d(\lambda,\sigma(a)) = |\lambda| \text{ for all } n \text{ and for all } \lambda.$$

Let $0 < \delta < 1$. Consider the set $M_\delta = D(0,\delta) \cup \{z \in \mathbb{C} : |z| = 3\}$. Let $b_\delta = \operatorname{diag}\{\alpha_i\}$ (as an operator over $l^2(\mathbb{N})$), where $\{\alpha_i\}$ is a countable dense set in M_δ. Further, $\sigma(b_\delta) = M_\delta$. Note that b_δ is a normal operator, hence it is of G_1-class and hence of G_n-class for all n. Thus,

$$\gamma_n(b_\delta,\lambda) = d(\lambda,\sigma(b_\delta))\ \forall n \in \mathbb{Z}_+ \cup \{0\} \text{ and } \forall\lambda \in \mathbb{C}.$$

Consider $c_\delta := a \oplus b_\delta = \begin{pmatrix} a & 0 \\ 0 & b_\delta \end{pmatrix}$. Thus c_δ is the direct sum of a which is not of G_n-class for any n and b_δ which is of G_n-class for every n. We shall establish that for small values of n, the behaviour of a decides that c_δ is not of G_n-class. On the other hand, for large values of n, the behaviour of b_δ decides that c_δ is of G_n-class. First, observe that $\sigma(c_\delta) = \sigma(a) \cup \sigma(b_\delta) = M_\delta$. Then

$$\gamma_n(c_\delta,\lambda) = \min\{\gamma_n(a,\lambda),\gamma_n(b_\delta,\lambda)\} = \begin{cases} 0 & \text{, if } |\lambda| \le \delta \\ |\lambda| - 3 & \text{, if } |\lambda| \ge 3 \\ \min\{|\lambda| - \delta, 3 - |\lambda|, \gamma_n(a,\lambda)\} & \text{, otherwise.} \end{cases}$$

Using 4 of Proposition 2.5, $\exists m \in \mathbb{N}$ such that for all $n \ge m$ we have,

$$|\lambda| - \delta \leq \gamma_n(a, \lambda) \leq |\lambda| + \delta \text{ for all } \delta \leq |\lambda| \leq 3.$$

So, for $n \geq m$,

$$\gamma_n(c_\delta, \lambda) = \min\{|\lambda| - \delta, 3 - |\lambda|\} = d(\lambda, \sigma(c_\delta)).$$

Thus, for $n \geq m$,

$$\gamma_n(c_\delta, \lambda) = d(\lambda, \sigma(c_\delta)) \ \forall \lambda \in \mathbb{C}.$$

For some small values of n and appropriate choice of λ and δ, we can show that c_δ will not belong to G_n-class. For example, choose $\delta = 0.1$, $|\lambda| = 1$ and $n = 0$. Then

$$\gamma_0(c_\delta) = \|(c_\delta - \lambda)^{-1}\|^{-1} = \min\{1 - 0.1, 3 - 1, \frac{1}{1+1}\} = 0.5 < 0.9 = d(\lambda, \sigma(b_\delta)).$$

So, for the above delta, c_δ is not of G_1-class. But already we have established that it is of G_n-class for sufficiently large n. So $G_n \subsetneq G_{n+1}$ for some $n \in \mathbb{N}$.

The following theorem extends the holomorphic functional calculus to the derivatives of an analytic map. See Theorem 7.11 of [3] for details.

Theorem 2.22. *Let A be a complex Banach algebra. Let $O \subseteq \mathbb{C}$ be an open neighbourhood of $\sigma(a)$ and C be a closed curve in O such that C surrounds $\sigma(a)$. Let f be analytic in O. We know from functional calculus:*

$$\tilde{f}(a) = \frac{1}{2\pi i} \int_C (z - a)^{-1} f(z) dz.$$

Let $m \in \mathbb{N}$. Let $g = f^{(m)}$, the mth derivative of f. Then g is holomorphic in O and

$$\tilde{g}(a) = \frac{m!}{2\pi i} \int_C (z - a)^{-(m+1)} f(z) dz.$$

Corollary 2.23. *Suppose A is a Banach algebra, $a \in A$ and $n \in \mathbb{Z}_+$. Then*

$$a = \lambda \iff \Lambda_{n,\epsilon}(a) = D(\lambda, \epsilon) \ \forall \epsilon > 0.$$

Proof. If $a = \lambda$, then by 1 of Theorem 2.8, $\Lambda_{n,\epsilon}(a) = D(\lambda, \epsilon)$ $\forall \epsilon > 0$. Conversely, in view of 6 of Theorem 2.8, without any loss of generality we can assume that $\lambda = 0$. Let C be the circle centred at 0 with radius ϵ. Clearly C encloses $\Lambda_{n,\epsilon}(a)$. Choose m such that $m + 1 = 2^n$. Let $f(z) = z^{m+1}$. Then $g(z) = f^{(m)}(z) = (m + 1)!z$. Also $M := \sup\{|f(z)| : z \in C\} = \epsilon^{m+1}$. Thus by Theorem 2.22,

$$\tilde{g}(a) = \frac{m!}{2\pi i} \int_C (z - a)^{-(m+1)} f(z) dz.$$

Since C is the boundary of $\Lambda_{n,\epsilon}(a)$, for all $z \in C$, $\|(z - a)^{-2^n}\|^{1/2^n} \leq \frac{1}{\epsilon}$, i.e., $\|(z - a)^{-(m+1)}\| \leq \frac{1}{\epsilon^{m+1}}$. Thus,

$$\|\tilde{g}(a)\| \leq \frac{m! M \times \text{length of } C}{2\pi \epsilon^{m+1}}.$$

So

$$\|\tilde{g}(a)\| = \|(m+1)!a\| \leq \frac{m!\epsilon^{m+1}2\pi\epsilon}{2\pi\epsilon^{m+1}}.$$

Thus $\|a\| \leq \frac{\epsilon}{m+1}$ $\forall \epsilon > 0$. Hence $a = 0$. $\square$

3. Topological properties of (n, ϵ)-pseudospectrum

In this section, we describe some topological properties of (n, ϵ)-pseudospectrum. Some of these properties (and also their proofs) are very similar to the corresponding properties of the pseudospectrum given in [14]. Also see [2], [8] and [20] for more details.

Theorem 3.1. *Let A be a Banach algebra and $a \in A$. Let $n \in \mathbb{Z}_+$. Define the map $F_a : \mathbb{R}^+ \to K(\mathbb{C})$ by $F_a(\epsilon) = \Lambda_{n,\epsilon}(a)$. Then F_a is right continuous.*

Proof. Let $\epsilon_0 \in \mathbb{R}^+$ and $\{\epsilon_k\}$ decrease to ϵ_0.

Claim. $\bigcap\limits_{k\in\mathbb{Z}_+} \Lambda_{n,\epsilon_k}(a) = \Lambda_{n,\epsilon_0}(a)$.

Note that

$$\begin{aligned} \lambda \in \bigcap_{k\in\mathbb{Z}_+} \Lambda_{n,\epsilon_k}(a) &\iff \gamma_n(a,\lambda) \leq \epsilon_k \ \forall k \\ &\iff \gamma_n(a,\lambda) \leq \epsilon_0 \\ &\iff \lambda \in \Lambda_{n,\epsilon_0}(a). \end{aligned}$$

Again, when $\{\epsilon_k\}$ decreases, $\{\Lambda_{n,\epsilon_k}(a)\}$ decreases and the latter converges to $\bigcap\limits_{k\in\mathbb{Z}_+} \Lambda_{n,\epsilon_k}(a)$ in Hausdorff metric, i.e., $d_H(\Lambda_{n,\epsilon_k}(a), \Lambda_{n,\epsilon_0}(a)) \to 0$ as $k \to \infty$. Thus F_a is right continuous. $\square$

Remark 3.2. Shargorodsky (see [20]) gave an example to show that the map $\epsilon \mapsto \Lambda_\epsilon(a)$ is not continuous. Using the same example, Krishnan and Kulkarni proved that the map $a \mapsto \Lambda_\epsilon(a)$ is not continuous (see [14]).

Theorem 3.3. *Let A be a Banach algebra. Let $a \in A$, $n \in \mathbb{Z}_+$ and $\epsilon > 0$. Then $\Lambda_{n,\epsilon}(a)$ has no isolated points.*

Proof. If $a = \lambda$, where $\lambda \in \mathbb{C}$, then $\Lambda_{n,\epsilon}(a) = D(\lambda, \epsilon)$ which has no isolated points. So assume $a \neq \lambda$ $\forall \lambda \in \mathbb{C}$. Suppose $\Lambda_{n,\epsilon}(a)$ has an isolated point μ. Then there exists an $r > 0$ such that $\forall \lambda$ with $0 < |\lambda - \mu| < r$, $\|(\lambda - a)^{-2^n}\|^{1/2^n} < \frac{1}{\epsilon}$.

Case 1: Suppose $\mu \in \Lambda_{n,\epsilon}(a) \setminus \sigma(a)$. By the Hahn–Banach theorem, $\exists f \in A'$ such that

$$f((\mu - a)^{-2^n}) = \|(\mu - a)^{-2^n}\| \text{ and } \|f\| = 1.$$

Let us define $g : \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$ by $g(z) = f((z - a)^{-2^n})$. Then g is analytic in $B(\mu, r)$. But $\forall \lambda$ with $0 < |\lambda - \mu| < r$, $|g(\lambda)| \leq \|(\lambda - a)^{-2^n}\| < \frac{1}{\epsilon^{2^n}}$ whereas

$$g(\mu) = f((\mu - a)^{-2^n}) = \|(\mu - a)^{-2^n}\| \geq \frac{1}{\epsilon^{2^n}}.$$

This contradicts the maximum modulus principle.

Case 2: Suppose $\mu \in \sigma(a)$. Letting $\lambda \to \mu$, $\|(\lambda - a)^{-2^n}\| \to \infty$. But for $0 < |\lambda - \mu| < r$, $\|(\lambda - a)^{-2^n}\| < \frac{1}{\epsilon^{2^n}}$, a contradiction. $\square$

Theorem 3.4. *Let A be a Banach algebra, $a \in A$, $n \in \mathbb{Z}_+$ and $\epsilon > 0$. Then $\Lambda_{n,\epsilon}(a)$ has a finite number of components and each component contains at least one element of $\sigma(a)$.*

Proof. From Theorem 2.8, we have, $\sigma(a) + D(0, \epsilon) \subseteq \Lambda_{n,\epsilon}(a)$. So $B(\lambda, \epsilon) \subseteq \Lambda_{n,\epsilon}(a)\ \forall \lambda \in \sigma(a)$. By compactness of $\sigma(a)$, $\exists \lambda_1, ..., \lambda_m$ such that $\sigma(a) \subseteq \bigcup\limits_{i=1}^{m} B(\lambda_i, \epsilon) \subseteq \Lambda_{n,\epsilon}(a)$. For each $i = 1, ..., m$, since $B(\lambda_i, \epsilon)$ is a connected subset of $\Lambda_{n,\epsilon}(a)$, $\exists$ a closed component C_i of $\Lambda_{n,\epsilon}(a)$ such that $B(\lambda_i, \epsilon) \subseteq C_i$. Thus $\sigma(a) \subseteq \bigcup\limits_{i=1}^{m} C_i \subseteq \Lambda_{n,\epsilon}(a)$.

Claim. $\Lambda_{n,\epsilon}(a) = \bigcup\limits_{i=1}^{m} C_i$.

If possible, let $\mu \in \Lambda_{n,\epsilon}(a) \setminus \bigcup\limits_{i=1}^{m} C_i$. Let $r > \|a\| + \epsilon$ and let $S := B(0, r) \setminus \bigcup\limits_{i=1}^{m} C_i$.

Then S is an open set containing μ. Let S_0 be the component of S containing μ. Since $\mathbb{C}$ is locally connected, S_0 is open. Again observe that $S_0 \subseteq \rho(a)$, the resolvent of a. Define $g : \rho(a) \to \mathbb{R}$ by $g(z) = \|(z - a)^{-2^n}\|$. The Hahn–Banach theorem guarantees the existence of an element $\phi \in A'$ such that

$$\phi((\mu - a)^{-2^n})) = \|(\mu - a)^{-2^n}\| \text{ and } \|\phi\| = 1.$$

Define $h : S \to \mathbb{C}$ by

$$h(z) = \phi((z - a)^{-2^n}).$$

Then h is analytic on S and

$$|h(z)| \leq \|(z - a)^{-2^n}\| = g(z)\ \forall z \in S \subseteq \rho(a).$$

Since $\Lambda_{n,\epsilon}(a) \subseteq B(0, r), \delta B(0, r) \subseteq \Lambda_{n,\epsilon}(a)^c$. Thus

$$g(z) = \|(z - a)^{-2^n}\| < \frac{1}{\epsilon^{2^n}}\ \forall z \in \delta B(0, r).$$

We assert that $g(z) = \frac{1}{\epsilon^{2^n}}\ \forall z \in \bigcup_{i=1}^{m} \delta C_i$. Since $\delta C_i \subseteq C_i\ \forall i = 1, ..., m$,

$$g(z) = \|(z - a)^{-2^n}\| \geq \frac{1}{\epsilon^{2^n}}\ \forall z \in \bigcup_{i=1}^{m} \delta C_i \subseteq \Lambda_{n,\epsilon}(a).$$

If $g(z) > \frac{1}{\epsilon^{2^n}}$, then $\exists$ a neighbourhood V of z such that

$$g(\lambda) > \frac{1}{\epsilon^{2^n}}\ \forall \lambda \in V.$$

Since each C_i a component of $\Lambda_{n,\epsilon}(a), \delta C_i \subseteq \Lambda_{n,\epsilon}(a)\ \forall i = 1, ..., m$. There must exist a point $z_0 \in V$ such that $z_0 \in \Lambda^c_{n,\epsilon}(a)$, i.e., $g(z_0) = \|(z_0 - a)^{-2^n}\| < \frac{1}{\epsilon^{2^n}}$, a contradiction. Hence our last assertion follows. Now $\delta S \subseteq \delta B(0, r) \cup \bigcup\limits_{i=1}^{m} \delta C_i$ and since S_0 is component of S, so $\delta S_0 \subseteq \delta S$.

Let $z \in \delta S_0$. Then $z \in \delta B(0, r)$ implies $|h(z)| \leq g(z) = \|(z - a)^{-2^n}\| < \frac{1}{\epsilon^{2^n}}$ whereas $z \in \bigcup\limits_{i=1}^{m} \delta C_i$ implies $|h(z)| \leq g(z) = \frac{1}{\epsilon^{2^n}}$. In any case, $|h(z)| \leq g(z) \leq \frac{1}{\epsilon^{2^n}}\ \forall z \in \delta S_0$.

Again, μ is an interior point of S_0 and

$$|h(\mu)| = |\phi((\mu - a)^{-2^n}))| = \|(\mu - a)^{-2^n})\| = h(\mu) \geq \frac{1}{\epsilon^{2^n}}.$$

By maximum-modulus principle, h is constant on S_0 and so

$$g(z) \geq |h(z)| = h(\mu) \geq \frac{1}{\epsilon^{2^n}} \ \forall z \in S_0.$$

Hence, $S_0 \subseteq \Lambda_{n,\epsilon}(a)$. Again, by continuity of h, we have

$$|h(z)| \geq \frac{1}{\epsilon^{2^n}} \ \forall z \in \overline{S_0}.$$

Now if $\delta S_0 \cap \delta B(0,r) \neq \varnothing$, then $\exists z \in \delta S_0 \cap \delta B(0,r)$ which leads a contradiction as $z \in \delta S_0$ gives $|h(z)| \geq \frac{1}{\epsilon^{2^n}}$ whereas $z \in \delta B(0,r)$ implies $|h(z)| \leq g(z) < \frac{1}{\epsilon^{2^n}}$. Again if $\delta S_0 \cap (\bigcup\limits_{i=1}^{m} \delta C_i) \neq \varnothing$, then $\exists j$ such that $\delta S_0 \cap \delta C_j \neq \varnothing$ and hence $S_0 \cup C_j$ becomes a connected component of $\Lambda_{n,\epsilon}(a)$. But C_j is a component of $\Lambda_{n,\epsilon}(a)$. Hence $S_0 \subseteq C_j$, a contradiction to the fact that $S_0 \subseteq B(0,r) \setminus \bigcup\limits_{i=1}^{m} C_i$. Thus $\delta S_0 = \varnothing$, a contradiction. Hence $\Lambda_{n,\epsilon}(a) = \bigcup\limits_{i=1}^{m} C_i$ and each C_i obviously contains a point of $\sigma(a)$. $\square$

The above theorem is useful in the computation of (n,ϵ)-pseudospectrum of the operator A_δ in Example (1). We also use this example to illustrate some results proved earlier.

Example 3.5. Let us consider the operator A_δ in Example (1). First, let us recall that

$$\sigma(A_\delta) + D(0,\epsilon) \subseteq \Lambda_{n,\epsilon}(A_\delta) \subseteq D(0, \|A_\delta\| + \epsilon) \text{ [by Theorem 2.8]}.$$

We note that for $0 < \delta \leq 1$, $\|A_\delta^{-m}\| = \frac{1}{\delta} \ \forall m \in \mathbb{N}$. Thus for $0 < \delta \leq \epsilon \leq 1$, we have

$$\|A_\delta^{-2^n}\|^{1/2^n} = \frac{1}{\delta^{1/2^n}} \geq \frac{1}{\epsilon^{1/2^n}}.$$

Hence $0 \in \Lambda_{n,\epsilon^{1/2^n}}(A_\delta)$. Since for $\delta > 0$, $\sigma(A_\delta)$ is the circle $\{z \in \mathbb{C} : |z| = 1\}$, using Theorem 3.4, $\Lambda_{n,\epsilon}(A_\delta)$ must have a component (path) containing the origin and a point on the circle. Let $z = re^{i\theta}$ where $|r| < 1$. There exists $z_0 = re^{i\theta_0} \in \Lambda_{n,\epsilon^{1/2^n}}(A_\delta)$. Consider the element $\lambda = e^{i(\theta - \theta_0)} \in \sigma(A_\delta)$. Since A_δ is a weighted shift, A_δ and λA_δ are unitarily equivalent (see [9]). It can be easily verified that they have the same (n,ϵ)-pseudospectrum. Then $\lambda z_0 \in \Lambda_{n,\epsilon^{1/2^n}}(A_\delta)$ and so $z = re^{i\theta} \in \Lambda_{n,\epsilon^{1/2^n}}(A_\delta)$.

We observe that for $0 < \delta \leq \epsilon < 1$, $\Lambda_{n,\epsilon^{1/2^n}}(A_\delta) = \{z : |z| \leq 1 + \epsilon^{1/2^n}\} \neq \sigma(A_\delta) + D(0, \epsilon^{1/2^n})$.

Thus, for $0 < \delta < 1$, A_δ does not belong to G_n-class for any n.

Also we have, in particular, $\Lambda_\epsilon(A_\delta) = \{z : |z| \leq 1 + \epsilon\}$ for $0 < \delta \leq \epsilon \leq 1$. Again, since $\sigma(A_0) = D(0,1)$, we have

$$D(0, 1+\epsilon) = \sigma(A_0) + D(0,\epsilon) \subseteq \Lambda_{n,\epsilon}(A_0) \subseteq D(0, \|A_0\| + \epsilon) = D(0, 1+\epsilon).$$

Thus, $\Lambda_{n,\epsilon}(A_0) = D(0, 1+\epsilon) \ \forall n \in \mathbb{Z}_+$.

We claim that $d_H(\Lambda_{n,\epsilon}(A_\delta), \Lambda_{n,\epsilon}(A_0)) \to 0$ as $\delta \to 0$.

Suppose $\epsilon > 0$ and $n \in \mathbb{Z}_+$ are given. Then, for $0 < \delta \leq 1$,

$$\|A_\delta^{-2^n}\|^{1/2^n} = \frac{1}{\delta^{1/2^n}} \geq \frac{1}{\epsilon} \iff \delta^{1/2^n} \leq \epsilon \iff \delta \leq \epsilon^{2^n}.$$

Choose $0 < \delta \leq \min\{\epsilon^{2^n}, 1\}$. Then $0 \in \Lambda_{n,\epsilon}(A_\delta)$. By similar argument as above, $\Lambda_{n,\epsilon}(A_\delta) = D(0, 1+\epsilon)$ and our claim easily follows. Thus the simple computation shows that $\Lambda_{n,\epsilon}(A_\delta)$ behaves better than $\sigma(A_\delta)$ as $\delta \to 0$.

In [11], using MATLAB Hansen numerically computed and pictorially described the (n, ϵ)-pseudospectrum of the above operator for some particular choices of n, ϵ. Recall from the above paragraph that if $0 < \delta \leq \min\{\epsilon^{2^n}, 1\}$, then $\Lambda_{n,\epsilon}(A_\delta) = D(0, 1+\epsilon)$. In Figure 4($b$) of [11], $n = 2, \epsilon = 0.025$ and $\delta = 10^{-16} < \epsilon^4$.

Next consider the case $0 < \epsilon < \delta \leq 1$. Note that, when $\delta = 1$, A_1 is the bilateral left shift which is a unitary operator and hence it is of G_1-class. So

$$\Lambda_{n,\epsilon}(A_1) = \Lambda_\epsilon(A_1) = \sigma(A_1) + D(0, \epsilon) = \{z \in \mathbb{C} : 1 - \epsilon \leq |z| \leq 1 + \epsilon\}.$$

Also, for any δ such that $0 < \epsilon < \delta < 1$, we have, by 8 of Theorem 2.8,

$$\begin{aligned}\{z \in \mathbb{C} : 1 - \epsilon \leq |z| \leq 1 + \epsilon\} = \Lambda_{n,\epsilon}(A_1) \subseteq \Lambda_{n,\epsilon}(A_\delta) \subseteq \Lambda_\epsilon(A_\delta) &\subseteq \{z \in \mathbb{C} : \frac{1}{\|A_\delta^{-1}\|} - \epsilon \leq |z| \leq 1 + \epsilon\} \\ &= \{z \in \mathbb{C} : \delta - \epsilon \leq |z| \leq 1 + \epsilon\}.\end{aligned}$$

Thus, $d_H(\Lambda_{n,\epsilon}(A_\delta), \Lambda_{n,\epsilon}(A_1)) \leq 1 - \delta \to 0$ as $\delta \to 1$. Again this shows that $\Lambda_{n,\epsilon}(A_\delta)$ behaves better than $\sigma(A_\delta)$ as $\delta \to 1$.

Let $0 < \epsilon < 1$ and $0 < \delta < 1$. For $|\lambda| < 1$, we have the following relation:

$$(\lambda I - A_\delta)^{-1} = -\sum_{k=0}^{\infty} \lambda^k A_\delta^{-k-1}. \tag{4}$$

Note that the right hand side of the above expression converges absolutely as $|\lambda| < 1$ and $\|A_\delta^{-k-1}\| = \frac{1}{\delta}\ \forall k$. Suppose $m \in \mathbb{N}$ such that $m + 1 = 2^n$. Differentiating the Equation (4), m times w.r.t. λ,

$$(-1)^m m!(\lambda I - A_\delta)^{-(m+1)} = \sum_{k=m}^{\infty} k(k-1)...(k-m+1)\lambda^{k-m} A_\delta^{-k-1}.$$

Hence,

$$\begin{aligned}\|(\lambda I - A_\delta)^{-(m+1)}\| &\leq \frac{1}{\delta} \sum_{k=0}^{\infty} \frac{(k+m)(k+m-1)...(k+1)}{m!} |\lambda|^k \\ &= \frac{1}{\delta(1 - |\lambda|)^{m+1}}.\end{aligned}$$

If such $\lambda \in \Lambda_{n,\epsilon}(A_\delta)$, then

$$\frac{1}{\epsilon^{2^n}} \leq \|(\lambda I - A_\delta)^{-2^n}\| \leq \frac{1}{\delta(1 - |\lambda|)^{2^n}}.$$

In that case, we have, $\delta(1 - |\lambda|)^{2^n} \leq \epsilon^{2^n}$, i.e., $1 - \frac{\epsilon}{\delta^{1/2^n}} \leq |\lambda|$. So, for any n,

$$\{z \in \mathbb{C} : 1 - \epsilon \leq |z| \leq 1 + \epsilon\} = \sigma(A_\delta) + D(0, \epsilon) \subseteq \Lambda_{n,\epsilon}(A_\delta) \subseteq \{z \in \mathbb{C} : 1 - \frac{\epsilon}{\delta^{1/2^n}} \leq |z| \leq 1 + \epsilon\}. \tag{5}$$

Thus

$$d_H(\Lambda_{n,\epsilon}(A_\delta), \sigma(A_\delta) + D(0, \epsilon)) \leq \frac{\epsilon}{\delta^{1/2^n}} - \epsilon \to 0$$

as $n \to \infty$. This illustrates the result 3 in Theorem 2.8.

Taking $\epsilon = 0.025$, $\delta = 0.005$ and $n = 1$, we get from Equation (5),

$$\{z \in \mathbb{C} : 0.975 \leq |z| \leq 1.025\} \subseteq \Lambda_{1,\epsilon}(A_\delta) \subseteq \{z \in \mathbb{C} : 0.6464 \leq |z| \leq 1.025\}.$$

This may be compared with Fig 4(c) of [11]. Again, taking $n = 2$, we get

$$\{z \in \mathbb{C} : 0.975 \leq |z| \leq 1.025\} \leq \Lambda_{n,\epsilon}(A_\delta) \subseteq \{z \in \mathbb{C} : 0.9060 \leq |z| \leq 1.025\}.$$

This can be compared with Fig. 4(d) of [11].

Remark 3.6. In the above example, we have shown that $d_H(\Lambda_{n,\epsilon}(A_\delta), \Lambda_{n,\epsilon}(A_0)) \to 0$ as $\delta \to 0$, that is, as $A_\delta \to A_0$ and $d_H(\Lambda_{n,\epsilon}(A_\delta), \Lambda_{n,\epsilon}(A_1)) \to 0$ as $\delta \to 1$, that is, as $A_\delta \to A_1$. This raises a natural question: given a sequence $\{a_k\}$ of elements in a Banach algebra A converging to an element $a \in A$ and $\epsilon > 0$, when can we conclude that $d_H(\Lambda_{n,\epsilon}(a_k), \Lambda_{n,\epsilon}(a)) \to 0$? This is equivalent to asking whether the map $a \mapsto \Lambda_{n,\epsilon}(a)$ is continuous. This will be investigated in detail in future. However, the continuity of the above map is known for bounded operators on a Banach space X when either X or X' is complex uniformly convex (e.g., Hilbert spaces, L^p spaces with $1 \leq p \leq \infty$) (see [2], [21] and the references cited there).

Acknowledgments

The first author would like to thank Arundhathi Krishnan and Markus Seidel for useful discussions. The first author would also like to thank the Department of Atomic Energy (DAE), India (Ref. No.: 2/39(2)/2015/NBHM/R&D-II/7440) for the financial support. The authors thank the referee for his/her comments that improved the presentation of the text.

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