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Some properties of unbounded operators with closed range

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Abstract. Let H_1, H_2 be Hilbert spaces and T be a closed linear operator defined on a dense subspace $D(T)$ in H_1 and taking values in H_2. In this article we prove the following results:

(i) Range of T is closed if and only if 0 is not an accumulation point of the spectrum $\sigma(T^*T)$ of T^*T,
In addition, if $H_1 = H_2$ and T is self-adjoint, then
(ii) $\inf\{\|Tx\|: x \in D(T) \cap N(T)^{\perp} \|x\| = 1\} = \inf\{|\lambda|: 0 \neq \lambda \in \sigma(T)\}$,
(iii) Every isolated spectral value of T is an eigenvalue of T,
(iv) Range of T is closed if and only if 0 is not an accumulation point of the spectrum $\sigma(T)$ of T,
(v) $\sigma(T)$ bounded implies T is bounded.

We prove all the above results without using the spectral theorem. Also, we give examples to illustrate all the above results.

Keywords. Densely defined operator; closed operator; Moore–Penrose inverse; reduced minimum modulus.

1. Introduction

Let H_1, H_2 be Hilbert spaces and T be a closed linear operator defined on a dense subspace $D(T)$ in H_1 and taking values in H_2. In this note we establish the following:

(i) Range of T is closed if and only if 0 is not an accumulation point of the spectrum $\sigma(T^*T)$ of T^*T,
In addition, if $H_1 = H_2$ and T is self-adjoint, then
(ii) Every isolated spectral value of T is an eigenvalue of T,
(iii) $\inf\{\|Tx\|: x \in D(T) \cap N(T)^{\perp}, \|x\| = 1\} = \inf\{|\lambda|: 0 \neq \lambda \in \sigma(T)\}$,
(iv) Range of T is closed if and only if 0 is not an accumulation point of the spectrum $\sigma(T)$ of T,
(v) $\sigma(T)$ bounded implies T is bounded.

Analogues of (i) and (ii) in the case of a bounded operator T are well-known and their proofs can be found in [10]. In fact, for a bounded operator T, [10] contains three proofs of (i) using different concepts in basic operator theory. It is to be mentioned that, all the results (i)–(v) are particularly important in the context of solving operator equations of the form $Tx = y$ (see [6, 7, 10] for elaboration of this theme).

It is well-known that, given any densely defined closed (possibly unbounded) operator T, the operators $\check{T} := (I + T^*T)^{-1}$ and $\hat{T} := (I + TT^*)^{-1}$ are defined on all of H_1 and H_2 respectively, and are bounded operators (cf. [14, 15]). Methods of our proofs for (i) and (ii) consist of applying the above mentioned results on bounded operators to the operators $\hat{T}$ and $\check{T}$, and then using certain relationships between T, $\check{T}$ and $\hat{T}$ proved in Proposition 2.7 of [7].

We found that the statement (iv) appeared in a paper of Beutler (Theorem 13, page 490 of [2]) for normal operators, without proof. Here we give a proof of this statement for self-adjoint operators using elementary techniques which does not involve even the spectral theorem. In fact, we prove all results without using the spectral theorem.

Since we are dealing with densely defined closed linear operators, we list below a few features of such operators which distinguish them from bounded operators.

(a) The domain of a closed unbounded operator is a proper subspace of the whole space.
(b) The spectrum of a bounded operator is a nonempty compact subset of the complex plane $\mathbb{C}$, whereas the spectrum of an unbounded operator can be an empty set or whole of the complex plane $\mathbb{C}$ or an unbounded closed subset of the complex plane $\mathbb{C}$ (cf. Example 5, page 254 of [13]).
(c) If T is a bounded linear operator and if $x_n \to x$ in H_1, then $Tx_n \to Tx$. On the other hand, if T is a closed operator, the convergence of Tx_n is not guaranteed but it is guaranteed that whenever Tx_n converges to some y, then $x \in D(T)$ and $y = Tx$.

This paper is organized as follows: In the second section we introduce notations and consider a few preliminary results which are useful to prove the main results. The third section contains proof of (i) and some spectral properties of positive operators. In the fourth section the statements (ii), (iii), (iv) and (v) are proved. In the fifth section we give some examples which illustrate our results.

2. Notations and preliminaries

Throughout the paper we denote Hilbert spaces over the field of complex numbers $\mathbb{C}$ by $H,\ H_1,\ H_2,\ H_3$, and inner product and the corresponding norm on a Hilbert space are denoted by $\langle .\ ,\ . \rangle$ and $\|\cdot\|$ respectively. Let us fix some more notations.

$\mathcal{L}(H_1, H_2)$: The set of all linear operators T with domain and range subspaces of H_1 and H_2 respectively.

Let $T \in \mathcal{L}(H_1, H_2)$. Then domain of T, range of T and null space of T are denoted by $D(T),\ R(T)$ and $N(T)$ respectively.

For $T \in \mathcal{L}(H_1, H_2)$ and $S \in \mathcal{L}(H_2, H_3)$ with $R(T) \subseteq D(S)$, we define the operator $ST \in L(H_1, H_3)$ by $ST(x) = S(Tx)$ for all $x \in D(T)$.

For $T \in \mathcal{L}(H_1, H_2)$ and $S \in \mathcal{L}(H_1, H_2)$, we define the operator $T + S \in \mathcal{L}(H_1, H_2)$ with domain as $D(T) \cap D(S)$ by $(T + S)(x) = Tx + Sx$ for all $x \in D(T) \cap D(S)$.

$\mathcal{B}(H_1, H_2)$: The space of all bounded (linear) operators from H_1 into H_2.

$\mathcal{C}(H_1, H_2)$: Set of all closed linear operators with $D(T) \subseteq H_1$ and $R(T) \subseteq H_2$.

$\mathcal{L}(H) := \mathcal{L}(H, H)$.

$\mathcal{C}(H) := \mathcal{C}(H, H)$.

$\mathcal{B}(H) := \mathcal{B}(H, H)$.

For a closed subspace M of H, P_M denotes the orthogonal projection with range M.

Note 2.1. By the closed graph theorem (cf. Theorem 7.1, page 231 of [11]), it follows that a closed operator $T \in \mathcal{C}(H_1, H_2)$ with $D(T) = H_1$ is bounded.

If S and T are two operators, then by $S \subseteq T$ we mean that S is the restriction of T to $D(S)$ i.e., $D(S) \subseteq D(T)$ and $Sx = Tx$, for all $x \in D(S)$, and in that case, we may also write S as $T|_{D(S)}$.

Suppose X_1 and X_2 are subspaces of a Hilbert space with $X_1 \cap X_2 = \{0\}$. Then we use the notation $X_1 \bigoplus X_2$ to denote the direct sum of X_1 and X_2, and $X_1 \bigoplus^{\perp} X_2$ to denote the orthogonal direct sum of X_1 and X_2 whenever $\langle x, y\rangle = 0$ for every $x \in X_1$ and $y \in X_2$.

Now, a few standard definitions.

DEFINITION 2.2

An operator $T \in \mathcal{L}(H_1, H_2)$ with domain $D(T)$ is said to be *densely defined* if $\overline{D(T)} = H_1$.

It is known that every densely defined operator $T \in \mathcal{C}(H_1, H_2)$ has a unique *adjoint* in $\mathcal{C}(H_2, H_1)$, that is, there exists a unique $T^* \in \mathcal{C}(H_2, H_1)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in D(T)$, $y \in D(T^*)$.

DEFINITION 2.3 [15]

A densely defined operator $T \in \mathcal{L}(H)$ is said to be *self-adjoint* if $D(T) = D(T^*)$ and $T^* = T$.

DEFINITION 2.4 [15]

A self-adjoint operator T is said to be *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in D(T)$.

DEFINITION 2.5 [4, 5, 14]

Let $T \in \mathcal{L}(H)$. If T is one to one, then the *inverse* of T is the linear operator $T^{-1}\colon R(T) \to H$ defined $T^{-1}(Tx) = x$ for all $x \in D(T)$. It can be seen that $TT^{-1}y = y$ for all $y \in R(T)$.

DEFINITION 2.6 [4, 15]

For $T \in \mathcal{L}(H)$, the *resolvent* of T is denoted by $\rho(T)$ and is defined as

$$\rho(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \text{ is bijective and } (T - \lambda I)^{-1} \in \mathcal{B}(H)\}.$$

DEFINITION 2.7 [4, 15]

For $T \in \mathcal{L}(H)$, the *spectrum* $\sigma(T)$, *approximate point spectrum* $\sigma_{\mathrm{a}}(T)$, and the *point spectrum* $\sigma_{\mathrm{p}}(T)$ are defined by

$$\sigma(T) = \mathbb{C}\backslash\rho(T),$$
$$\sigma_{\mathrm{a}}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \text{ is not bounded below}\},$$
$$\sigma_{\mathrm{p}}(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \text{ is not one to one}\}$$

respectively.

By the closed graph theorem (cf. Theorem 7.1, page 231 of [11]), it follows that if $T \in \mathcal{C}(H)$, then

$$\sigma(T) = \{\lambda \in \mathbb{C}\colon T - \lambda I \text{ is not bijective}\}.$$

DEFINITION 2.8 [1]

Let $T \in \mathcal{L}(H_1, H_2)$. The subspace $C(T) := D(T) \cap N(T)^{\perp}$ is called the *carrier* of T. We denote the operator $T|_{C(T)}$ by T_0.

Note 2.9 (*Moore–Penrose inverse*). Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then there exists a unique closed densely defined operator $T^{\dagger}$ with domain $D(T^{\dagger}) = R(T) \bigoplus^{\perp} R(T)^{\perp}$ and codomain $C(T)$ satisfying the following properties:

(1) $N(T^{\dagger}) = R(T)^{\perp}$,
(2) $T^{\dagger}Tx = P_{\overline{R(T^{\dagger})}}\, x$ for all $x \in D(T)$,
(3) $TT^{\dagger}y = P_{\overline{R(T)}}\, y$ for all $y \in D(T^{\dagger})$.

This unique operator $T^{\dagger}$ is said to be the Moore–Penrose inverse of T. (Recall that for a closed subspace M of Hilbert space, P_M denotes the orthogonal projection with range M).

The following property of $T^{\dagger}$ is also well-known.

For every $y \in D(T^{\dagger})$, let $L(y) := \{x \in D(T): \|Tx - y\| \leq \|Tu - y\| \;\; \forall u \in D(T)\}$. Then $T^{\dagger}y \in L(y)$ and $\|T^{\dagger}y\| \leq \|x\| \;\; \forall\, x \in L(y)$.

A different treatment of $T^{\dagger}$ is described in [11] (pages 336, 339, 340), where the authors call it 'the maximal Tseng inverse'.

It can be seen that if $T \in \mathcal{C}(H_1, H_2)$ is injective, then for every $y \in R(T)$, $T^{\dagger}y = T^{-1}y$.

DEFINITION 2.10 [1, 9]

Let $T \in \mathcal{C}(H_1, H_2)$. The *reduced minimum modulus* of T is defined by $\gamma(T) := \inf\{\|Tx\|: x \in C(T),\; \|x\| = 1\}$.

Some known properties of $\gamma(T)$ which are listed in the following proposition are extensively used in due course.

PROPOSITION 2.11 [1, 9]

For a densely defined $T \in \mathcal{C}(H_1, H_2)$, the following statements (1) *to* (8) *are equivalent.*

(1) *$R(T)$ is closed.*
(2) *$R(T^*)$ is closed.*
(3) *$T_0 := T|_{C(T)}$ has a bounded inverse.*
(4) $\gamma(T) > 0$.
(5) *$T^{\dagger}$ is bounded.*
(6) $\gamma(T) = \gamma(T^*)$.
(7) *$R(T^*T)$ is closed.*
(8) *$R(TT^*)$ is closed.*

Now we prove two more properties of $\gamma(T)$.

PROPOSITION 2.12

Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then we have the following:

(1) *If $R(T)$ is closed, then* $\gamma(T) = \frac{1}{\|T^{\dagger}\|}$.
(2) $\gamma(T^*T) = \gamma(T)^2$.

Proof.

Proof of (1). The proof for the case of bounded operators is known (see for e.g., [1, 3, 12]). In the case of a closed operator also, the proof goes along similar lines. For the sake of completeness, we observe that

$$\begin{aligned}\|T^\dagger\| &= \sup\left\{\frac{\|T^\dagger y\|}{\|y\|}: 0 \neq y \in D(T^\dagger)\right\}\\ &= \sup\left\{\frac{\|T^\dagger y\|}{\|y\|}: 0 \neq y \in R(T)\right\}\\ &= \sup\left\{\frac{\|x\|}{\|Tx\|}: 0 \neq x \in C(T)\right\}\\ &= \left(\inf\left\{\frac{\|Tx\|}{\|x\|}: 0 \neq x \in C(T)\right\}\right)^{-1}\\ &= \gamma(T)^{-1}.\end{aligned}$$

Proof of (2). If $\gamma(T) = 0$, then by the equivalence of (1), (4) and (7) in Proposition 2.11, $R(T)$ is not closed and $\gamma(T^*T) = 0$. Next, let us assume that $\gamma(T) > 0$. Again, by the equivalence of (1), (4) and (5) in Proposition 2.11, $R(T)$ is closed and $T^\dagger$ is bounded, so that $T^{\dagger *}T^\dagger$ is also bounded. Hence by (1) and using the fact that $(T^*T)^\dagger = T^\dagger T^{*\dagger}$ (cf. Theorem 2, page 341 of [1]), we have

$$\gamma(T^*T) = \frac{1}{\|(T^*T)^\dagger\|} = \frac{1}{\|T^\dagger\|^2} = \gamma(T)^2.$$

This completes the proof. ∎

In the following proposition we list some well-known facts.

PROPOSITION 2.13 [1]

Let $T \in \mathcal{C}(H_1, H_2)$ be a densely defined operator. Then

(1) $N(T) = R(T^*)^\perp$
(2) $N(T^*) = R(T)^\perp$
(3) $N(T^*T) = N(T)$ *and*
(4) $\overline{R(T^*T)} = \overline{R(T^*)}$.

Now, we recall the following result proved in [7].

PROPOSITION 2.14 [7]

Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then $I + T^*T$ and $I + TT^*$ are bijective densely defined closed operators, and

$$\hat{T} = (I + TT^*)^{-1}, \quad \check{T} = (I + T^*T)^{-1}$$

have the following properties:

(1) $\hat{T}$ *and* $\check{T}$ *are bounded and self-adjoint.*
(2) $T^*\hat{T}$ *and* $T\check{T}$ *are bounded and positive operators.*
(3) $\hat{T}T \subseteq T\check{T}$ *and* $\check{T}T^* \subseteq T^*\hat{T}$.
(4) $\|I - \check{T}\| \leq 1$.
(5) $R(I - \check{T}) = R(T^*T)$.

Theorem 2.15. *For* $T \in \mathcal{C}(H)$, *we have the following*:

(1) *If* $\mu \in \mathbb{C}$ *and* $\lambda \in \sigma(T)$, *then* $\lambda + \mu \in \sigma(T + \mu I)$.
(2) *If* $\alpha \in \mathbb{C}$ *and* $\lambda \in \sigma(T)$, *then* $\alpha\lambda \in \sigma(\alpha T)$.
(3) $\sigma(T^2) = \{\lambda^2 \colon \lambda \in \sigma(T)\}$.

Proof. Proofs of (1) and (2) follow directly from the definition of the spectrum. The relation in (3) is proved in Theorem 9.6, page 326 of [16] using operational calculus. Here we present a proof by using elementary methods. For this, let $\lambda \in \mathbb{C}$ be such that $\lambda^2 \notin \sigma(T^2)$. Then $(T^2 - \lambda^2 I)^{-1} \in \mathcal{B}(H)$. By definition this means that there exists a unique operator $S_\lambda \in \mathcal{B}(H)$ such that $S_\lambda(T^2 - \lambda^2 I) = I|_{D(T^2)} \subseteq (T^2 - \lambda I)S_\lambda = I$. That is,

$$S_\lambda(T - \lambda I)(T + \lambda I) = I|_{D(T^2)} \subseteq (T - \lambda I)(T + \lambda I)S_\lambda = I.$$

From the last equality, it also follows that $T - \lambda I$ is onto.

Next, we show that $T - \lambda I$ is one to one. For this, let $x \in D(T)$ such that $(T - \lambda I)x = 0$, that is, $Tx = \lambda x$. Then we have $Tx \in D(T)$ so that $x \in D(T^2)$. Applying T to both sides of the equation $Tx = \lambda x$, we get $T^2 x = \lambda^2 x$, that is, $(T^2 - \lambda^2 I)x = 0$. Since $T^2 - \lambda^2 I$ is injective, we have $x = 0$. Thus, $T - \lambda I$ is one to one as well. Consequently, $\lambda \notin \sigma(T)$.

For the other way implication, let $\lambda \in \mathbb{C}$ be such that $\lambda^2 \in \sigma(T^2)$. We have to show that either $\lambda \in \sigma(T)$ or $-\lambda \in \sigma(T)$. Suppose this is not true. Then both $T - \lambda I$ and $T + \lambda I$ have bounded inverses. First we show that $N(T^2 - \lambda^2 I) = \{0\}$. For this, let $x \in D(T^2) \subseteq D(T)$ such that $(T^2 - \lambda^2 I)x = 0$, that is, $(T - \lambda I)(T + \lambda I)x = 0$. Now, by the injectivity of $T - \lambda I$ and $T + \lambda I$, it follows that $x = 0$. It remains to show that $T^2 - \lambda I$ is onto. For this, let $y \in H$. Since $T - \lambda I$ is onto, there exists $u \in D(T) \subseteq H$ such that $(T - \lambda I)u = y$. As $T + \lambda I$ is onto, there exists $x \in D(T)$ such that $(T + \lambda I)x = u$. Hence $(T - \lambda I)(T + \lambda I)x = y$, that is, $(T^2 - \lambda I)x = y$. This completes the proof of (3). ■

3. A spectral characterization of closed range operators

First we prove two preliminary results which are required for proving our main theorems. Recall that for $T \in \mathcal{L}(H)$, $T_0 := T|_{C(T)}$.

PROPOSITION 3.1

Let $T \in \mathcal{L}(H)$ *be a positive operator. Then the following results hold.*

(1) $T^\dagger$ *is positive.*
(2) $\sigma(T)\backslash\{0\} = \sigma(T_0)\backslash\{0\}$.
(3) $\sigma(T^\dagger)\backslash\{0\} = \sigma(T_0^{-1})\backslash\{0\}$.

(4) $\sigma(T) = \sigma_a(T)$.
(5) $0 \notin \sigma(I+T)$, *that is* $(I+T)^{-1} \in \mathcal{B}(H)$.
(6) *If* $\lambda > 0$, *then* $\lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^\dagger)$.
(7) *If* $0 \notin \sigma(T)$, *then* $0 \neq \lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{-1})$.

Proof.

Proof of (1). Let $T \in \mathcal{L}(H)$ be a positive operator. Then T is a self-adjoint operator as well. Using the relation $T^{\dagger *} = T^{*\dagger}$ (cf. Theorem 2, page 341 of [1]), it is easy to show that $T^\dagger$ is self-adjoint. Since $D(T^\dagger) = R(T) \bigoplus^\perp R(T)^\perp$ (see Theorem 2, page 341 of [1] for details), there exist $u \in C(T)$ and $v \in R(T)^\perp$ such that $y = Tu + v$. Using (2) and (4) in Note 2.9, and using the fact that $N(T)^\perp = \overline{R(T)}$ (cf. Proposition 2.13(1)), we have

$$\begin{aligned}\langle T^\dagger y, y\rangle &= \langle T^\dagger y, Tu + v\rangle = \langle T^\dagger y, Tu\rangle \\ &= \langle TT^\dagger y, u\rangle = \langle P|_{\overline{R(T)}}\, y, u\rangle = \langle Tu, u\rangle \geq 0.\end{aligned}$$

Proof of (2). Since T is self-adjoint, it is reducible by $N(T)$, that is,

$$T(D(T) \cap N(T) \subseteq N(T), \quad T(D(T) \cap N(T)^\perp) \subseteq N(T)^\perp.$$

Therefore, by a known result (cf. Theorem 5.4, page 289 of [16]),

$$\sigma(T) = (\sigma(T|_{N(T)}) \cup \sigma(T|_{C(T)}).$$

That is, $\sigma(T) = \{0\} \cup \sigma(T_0)$. Hence $\sigma(T)\backslash\{0\} = \sigma(T_0)\backslash\{0\}$.

Proof of (3). Since $T^\dagger$ is self-adjoint, it is reducible by $R(T)^\perp$. Now, using the fact that $T^\dagger|_{R(T)} = T_0^{-1}$, (2) implies that, $\sigma(T^\dagger)\backslash\{0\} = \sigma(T_0^{-1})\backslash\{0\}$.

Proof of (4). By definition $\sigma_a(T) \subseteq \sigma(T)$, and since T is self-adjoint, $\sigma(T) \subseteq \mathbb{R}$. Now, suppose $\lambda \notin \sigma_a(T)$. Then there exists a positive number k such that

$$\|Tx - \lambda x\| \geq k\|x\|, \text{ for all } x \in D(T).$$

This shows that $R(T - \lambda I)$ is closed and $T - \lambda I$ is one to one. As $T - \lambda I$ is self-adjoint, we also have $\overline{R(T - \lambda I)} = H$, by (2) in Proposition 2.13. Hence, $\lambda \notin \sigma(T)$.

Proof of (5). Since T is positive, $I+T$ is positive. Assume for a moment that $0 \in \sigma(I+T)$. Now by (5), $\sigma(I+T) = \sigma_a(I+T)$. Hence there exists a sequence $(x_n) \in D(T)$, with $\|x_n\| = 1$, for each n such that $\|(I+T)x_n\| \to 0$ as $n \to \infty$. Now using the Cauchy–Schwartz inequality, it follows that $\langle (I+T)x_n, x_n\rangle \to 0$ as $n \to \infty$. That is $\langle Tx_n, x_n\rangle \to -1$. On the other hand, since T is positive $\langle Tx_n, x_n\rangle \geq 0$ for all n. This is a contradiction. Therefore $0 \notin \sigma(I+T)$.

Proof of (6). Let $\lambda \in (0, \infty)$ be such that $\lambda \in \sigma(T)$. Then by (4), there exists a sequence $(x_n) \in C(T)$ with $\|x_n\| = 1$ for all n such that $\|Tx_n - \lambda x_n\| = \|T_0 x_n - \lambda x_n\| \to 0$ as $n \to \infty$. Let $y_n = T_0(x_n)$. Since $T_0\colon C(T) \to R(T)$ is bijective, $x_n = T_0^{-1}(y_n)$. Thus $\|y_n - \lambda T_0^{-1}(y_n)\| \to 0$. Also for large values of n, $\|y_n\|$ is close to $|\lambda|$ and hence non-zero. Now taking $z_n = y_n/\|y_n\|$, we have $\|T_0^{-1} z_n - \frac{1}{\lambda} z_n\| \to 0$ as $n \to \infty$. This together with (3), we have

$$\frac{1}{\lambda} \in \sigma(T_0^{-1}\backslash\{0\}) = \sigma(T^\dagger)\backslash\{0\}.$$

If $\frac{1}{\lambda} \in \sigma(T^{\dagger})$, then by the above argument it follows that $\frac{1}{1/\lambda} \in \sigma(T^{\dagger\dagger})$. But, it is known that $T^{\dagger\dagger} = T$ (see Theorem 2, page 341 of [1]). Thus, we have $\lambda \in \sigma(T)$.

Proof of (7). Since T^{-1} exists, we have $T^{-1} = T_0^{-1} = T^{\dagger}$. Hence, the result follows from (5). ■

PROPOSITION 3.2

Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Let $\check{T} = (I + T^*T)^{-1}$ and $A = I - \check{T}$. Then

(1) $\sigma(A) = \left\{\frac{\lambda}{1+\lambda} : \lambda \in \sigma(T^*T)\right\}$

(2) $\sigma(T^*T) = \left\{\frac{\mu}{1-\mu} : \mu \in \sigma(A)\right\}$.

Proof. By Theorem 2.15 and Proposition 3.1, we have $\mu \in \sigma(A)$ if and only if there exists $\lambda \in \sigma(T^*T)$ such that $\mu = 1 - \frac{1}{\lambda+1} = \frac{\lambda}{1+\lambda}$. Since $I - A = (I + T^*T)^{-1}$ is injective and $T^*T = A(I-A)^{-1}$ on $D(T^*T)$, again by Theorem 2.15 and Proposition 3.1, $\lambda \in \sigma(T^*T)$ if and only if there exists a $\mu \in \sigma(A)$ such that

$$\lambda = \frac{1}{1-\mu} - 1 = \frac{\mu}{1-\mu}.$$

This completes the proof. ■

We now give a proof of the statement (1) given in the Introduction.

Theorem 3.3. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then $R(T)$ is closed in H_2 if and only if there exists $r > 0$ such that $\sigma(T^*T) \subseteq \{0\} \cup [r, \infty)$.*

Proof. By Proposition 2.11 and Proposition 2.12, $R(T)$ is closed if and only if $R(T^*T)$ is closed. It can be easily seen that $R(T^*T) = R(A)$, where $A := I - (I + T^*T)^{-1}$. Note that A is a bounded self-adjoint operator on H_1. Hence $R(A)$ is closed if and only if 0 is not an accumulation point of $\sigma(A^*A) = \sigma(A^2)$ if and only if 0 is not an accumulation point of $\sigma(A)$ (cf. Theorem 2.5 of [10]). Now by Proposition 3.2, 0 is not an accumulation point of $\sigma(A)$ if and only if 0 is not an accumulation point of $\sigma(T^*T)$, equivalently, there exists an $r > 0$ such that $\sigma(T^*T) \subseteq \{0\} \cup [r, \infty)$. ■

In the remaining part of this section we prove some spectral properties of positive operators which will be used in the next section.

Theorem 3.4. *Let $T \in \mathcal{L}(H)$ be a positive operator. If $\sigma(T)$ is bounded, then T is bounded (In particular, $D(T) = H$).*

Proof. Since T is positive, by (5) of Proposition 3.1, $I + T$ is bijective and $(I + T)^{-1} \in \mathcal{B}(H)$. Let $A = T(I + T)^{-1}$. Clearly, A is a bounded operator. In fact, A is a positive operator. To see this, let $x \in H$ and $y := (I + T)^{-1}x$. Then we have

$$\begin{aligned}\langle Ax, x\rangle &= \langle T(I + T)^{-1}x, x\rangle \\ &= \langle Ty, x\rangle,\end{aligned}$$

$$= \langle Ty, (I+T)y\rangle$$

$$= \langle Ty, y\rangle + \langle Ty, Ty\rangle.$$

Thus, by positivity of T, $\langle Ax, x\rangle \geq 0$ for all $x \in H$ showing that A is positive as well. In particular, A is a bounded self-adjoint operator.

We note that $T = A(I - A)$. Now, since $\sigma(A) \neq \emptyset$, using the argument similar to the one used in proving Proposition 3.2, we have $T = A(I-A)^{-1}$ on $D(T)$ and $\sigma(T) = \{\lambda/(1-\lambda) : \lambda \in \sigma(A)\}$.

Now suppose that $\sigma(T)$ is bounded. Then there exists a $k > 0$ such that $\frac{\lambda}{1-\lambda} \leq k$ for every $\lambda \in \sigma(A)$. That is, $\lambda \leq \frac{k}{1+k} < 1$ for every $\lambda \in \sigma(A)$. Since A is bounded and self-adjoint operator, we have $\|A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\} \leq \frac{k}{1+k} < 1$. Thus $I - A$ is bijective and $(I-A)^{-1}$ is a bounded operator. Now, since T is closed, $D(T)$ is closed (cf. Theorem 3.17, page 165 of [11]). Hence $T \in \mathcal{B}(H)$. ■

Now we use Theorem 3.4 to prove the following result.

Theorem 3.5. *Let $T \in \mathcal{L}(H)$ be a positive operator and*

$$d(T) := \inf\{|\lambda| : \lambda \in \sigma(T)\backslash\{0\}\} = d(0, \sigma(T)\backslash\{0\}).$$

Then $\gamma(T) = d(T)$.

Proof. We consider the following two cases:

Case 1. $\gamma(T) > 0$. We know by Proposition 2.11 that if $\gamma(T) > 0$, then $R(T)$ is closed. In this case T_0^{-1} and $T^\dagger$ are bounded, self-adjoint operators with $\|T_0^{-1}\| = \|T^\dagger\| = 1/\gamma(T)$. Hence by Proposition 3.1,

$$\begin{aligned}
\gamma(T) &= \frac{1}{\|T^\dagger\|} \\
&= \frac{1}{\sup\{|\mu| : \mu \in \sigma(T^\dagger)\}} \\
&= \frac{1}{\sup\{|\mu| : \mu \in \sigma(T_0^{-1})\}} \\
&= \frac{1}{\sup\{1/|\lambda| : 0 \neq \lambda \in \sigma(T_0)\}} \\
&= \inf\{|\lambda| : 0 \neq \lambda \in \sigma(T_0)\} \\
&= d(T).
\end{aligned}$$

Case 2. $\gamma(T) = 0$. By Proposition 2.11, $T^\dagger$ is an unbounded operator. Hence, by Proposition 3.1 and by Theorem 3.4, $\sigma(T^\dagger)$ is bounded. Therefore, for each $n \in \mathbb{N}$, there exists $\lambda_n \in \sigma(T^\dagger)$ such that $\lambda_n \geq n$. Now by Proposition 3.1, $\frac{1}{\lambda_n} \in \sigma(T)$. Since, $\frac{1}{\lambda_n} \to 0$ as $n \to \infty$, it follows that $d(T) = 0$. ■

Theorem 3.6. *Suppose $T \in \mathcal{L}(H)$ is a positive operator and 0 is an isolated spectral value of T. Then $0 \in \sigma_{\mathrm{p}}(T)$.*

Proof. Since 0 is an isolated spectral value of T, $d(T) > 0$. Hence by Theorem 3.5, $\gamma(T) > 0$ so that by Proposition 2.11, $R(T)$ is closed. If $0 \notin \sigma_{\mathrm{p}}(T)$, then $N(T) = \{0\}$ so that we also have $R(T) = \overline{R(T)} = N(T)^{\perp} = H$, making T bijective and hence $0 \notin \sigma(T)$, a contradiction. Hence $0 \in \sigma_{\mathrm{p}}(T)$. ■

Remark 3.7. The converse of Theorem 3.6 need not be true. To see this, consider $T: \ell^2 \to \ell^2$ defined by

$$T(x_1, x_2, x_3, x_4, x_5, \dots) = \left(0, 2x_2, \frac{1}{3}x_3, 4x_4, \frac{1}{5}x_5, \dots\right),$$

where $D(T) = \left\{x \in \ell^2 : (0, 2x_2, \frac{1}{3}x_3, 4x_4, \frac{1}{5}x_5, \dots) \in \ell^2\right\}$. Here T is a positive operator. Since T is not one to one, $0 \in \sigma_{\mathrm{p}}(T)$ but it is not an isolated point of the spectrum $\sigma(T) = \left\{0, 2, \frac{1}{3}, 4, \frac{1}{5}, \dots\right\}$.

4. Self-adjoint operators

In this section we extend the results about the positive operators proved in the last section to self-adjoint operators. In the process, we give an elementary proof of a result of Beutler which was stated in (Theorem 13, page 490 of [2]) without proof. For proving this result we need the following.

Theorem 4.1. *Let $T \in \mathcal{L}(H)$ be a self-adjoint operator. Then every isolated spectral value of T is an eigenvalue.*

Proof. Let λ be an isolated point of $\sigma(T)$. Then 0 is an isolated point of $\sigma(T - \lambda I)$. Hence by Theorem 2.15, 0 is an isolated point of $\sigma(T - \lambda I)^2$. As $(T - \lambda I)^2$ is a positive operator, by Theorem 3.6, 0 is an eigenvalue of $(T - \lambda I)^2$. But $N(T - \lambda I) = N((T - \lambda I)^*(T - \lambda I)) = N(T - \lambda I)^2 \neq \{0\}$. Hence λ is an eigenvalue of T. ■

PROPOSITION 4.2

Let $T \in \mathcal{L}(H)$ be self-adjoint. Then the following statements hold.

(1) *$T^{\dagger}$ is self-adjoint.*
(2) $\sigma(T)\backslash\{0\} = \sigma(T_0)\backslash\{0\}$.
(3) $\sigma(T^{\dagger})\backslash\{0\} = \sigma(T_0^{-1})\backslash\{0\}$.
(4) *Let $0 \neq \lambda \in [0, \infty)$. Then $\lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{\dagger})$.*
(5) $\sigma(T) = \sigma_a(T))$.
(6) *If T^{-1} exists, then $0 \neq \lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{-1})$.*

Proof. The proofs of all these statements are analogous to those of Proposition 3.1. ■

Theorem 4.3. *Let T be self-adjoint and*

$$d(T) = \inf\{|\lambda| : \lambda \in \sigma(T)\backslash\{0\}\} = d(0, \sigma(T)\backslash\{0\}).$$

Then $\gamma(T) = d(T)$.

Proof. Again note that T^2 is a positive operator. Hence $\gamma(T^2) = d(T^2)$ by Theorem 3.5. But $\gamma(T^2) = \gamma(T^*T) = \gamma(T)^2$ by Proposition 2.12. Also $d(T^2) = d(T)^2$ by Theorem 2.15. ■

Theorem 4.4. *Let $T \in \mathcal{L}(H)$ be self-adjoint. Then $R(T)$ is closed if and only if* 0 *is not an accumulation point of $\sigma(T)$.*

Proof. By Proposition 2.11, $R(T)$ is closed if and only if $\gamma(T) > 0$ and by Theorem 4.3, $\gamma(T) = d(T)$. Hence, $R(T)$ is closed if and only if $d(T) > 0$ if and only if 0 is not an accumulation point of $\sigma(T)$. ■

Remark 4.5. Theorem 4.4 can be used to give another proof of Theorem 3.3 as follows:

Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then $R(T)$ is closed in H_2 if and only if $R(T^*T)$ is closed (Proposition 2.11) if and only if 0 is not an accumulation point of $\sigma(T^*T)$ by Theorem 4.4.

Theorem 4.6. *Let T be a self-adjoint operator. If $\sigma(T)$ is bounded, then $T \in \mathcal{B}(H)$.*

Proof. Note that T^2 is positive and $\sigma(T^2) = \{\lambda^2 : \lambda \in \sigma(T)\}$ is bounded. Hence by Theorem 3.4, T^2 is bounded with domain $D(T^2) = H$. Hence $D(T) = H$, by the closed graph theorem. ■

For the next Corollary, we recall that for $T \in \mathcal{B}(H)$, $|T| \in \mathcal{B}(H)$ is the unique postiive operator satisfying $|T|^2 = T^*T$. A simple proof for the existence of such operator $|T|$ can be found in ([8], Theorem 5.1.3, page 177).

COROLLARY 4.7

Let $T \in \mathcal{B}(H)$. Then

$$\gamma(T) = \inf\{\lambda : \lambda \in \sigma(|T|)\backslash\{0\}\},$$

where $|T|$ denotes the square root of T^*T.

Proof. The operator $|T|$ is bounded, self-adjoint and positive. Using Proposition 2.12, we have

$$\gamma(|T|)^2 = \gamma(|T|^2) = \gamma(T^*T) = \gamma(T)^2.$$

Now by Theorem 4.3, we get the result. ■

5. Examples

Example 5.1. Let $H = \ell^2$. Define T on H by

$$T(x_1, x_2, x_3, \ldots, x_n, \ldots) = (x_1, 2x_2, 3x_3, \ldots, nx_n, \ldots)$$

whose domain is

$$D(T) = \left\{(x_1, x_2, x_3, \ldots, x_n, \ldots) \in H : \sum_{j=1}^{\infty} |jx_j|^2 < \infty\right\}.$$

Clearly T is unbounded and closed since $T^* = T$. And since $D(T)$ contains c_{00}, the space of all sequences having atmost finitely many non-zero terms, we have $\overline{D(T)} = H$. Also $R(T)$ is closed (see Example 3.12, page 168 of [11]). Here $\gamma(T) = 1$. Now $\sigma_{\mathrm{p}}(T) = \{n: n \in \mathbb{N}\}$. We claim that $\sigma(T) = \sigma_{\mathrm{p}}(T)$. Suppose $\lambda \neq n$, for all $n \in \mathbb{N}$. Then there exists an $\eta > 0$ such that $|\lambda - n| \geq \eta$, for all $n \in \mathbb{N}$. Define $S_\lambda\colon H \to H$ by $S_\lambda(x) = \left(\frac{x_1}{\lambda-1}, \ldots, \frac{x_n}{\lambda-n}, \ldots\right)$. Then S_λ is bounded, $\|S_\lambda\| \leq \frac{1}{\eta}$ and S_λ is the inverse of $\lambda I - T$. Here $d(T) = 1$. Similarly we can prove that $\sigma(T^*T) = \sigma(T^2) = \{n^2: n \in \mathbb{N}\}$. This illustrates Theorems 3.3, 4.1, 4.3, 4.4 and 4.6.

Example 5.2. On ℓ^2, define an operator T by

$$T(x_1, x_2, \ldots, x_n, \ldots,) = (0, 2x_2, 3x_3, 4x_4, \ldots,)$$

with

$$D(T) = \left\{(x_1, x_2, x_3, \ldots, x_n, \ldots)\colon \sum_{j=2}^{\infty} |jx_j|^2 < \infty\right\}.$$

Here $T = T^*$, T is densely defined and closed, $N(T) = \{(x_1, 0, 0, \ldots)\colon x_1 \in \mathbb{C}\}$ and

$$C(T) = \left\{(0, x_2, x_3, \ldots)\colon \sum_{j=2}^{\infty} |jx_j|^2 < \infty\right\},$$

$$\check{T}(x_1, x_2, x_3, \ldots, x_n, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{5}, \ldots \frac{x_n}{(n-1)^2+1}, \ldots\right).$$

Note that $\sigma(T) = \{n-1: n \in \mathbb{N}\}$, $\sigma(T^*T) = \{(n-1)^2: n \in \mathbb{N}\}$ and $\gamma(T) = 1 = d(T)$. Also $\|Tx\| \geq \|x\|$ for all $x \in C(T)$. This illustrates Theorems 3.3, 4.1, 4.3, 4.4 and 4.6.

Example 5.3. Let $H = \ell^2$. Define an operator T on H by

$$T(x_1, x_2, x_3, \ldots, x_n, \ldots) = \left(x_1, 2x_2, \frac{1}{3}x_3, 4x_4, \frac{1}{5}x_5, \ldots\right)$$

with domain

$$D(T) = \left\{(x_1, x_2, x_3, \ldots, x_n, \ldots)\colon \left(x_1, 2x_2, \frac{1}{3}x_3, 4x_4, \frac{1}{5}x_5, \ldots\right) \in H\right\}.$$

This operator is densely defined since its domain contains c_{00}. Moreover $T^* = T$, so that T is closed. We can easily show that $N(T) = N(T^*) = \{0\}$. Hence $C(T) = D(T)$. This implies that $\overline{R(T)} = H$. We show that $R(T)$ is a proper dense subspace. Now consider $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right) \in H$. We show that this is not in $R(T)$. Suppose $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right) = \left(x_1, 2x_2, \frac{1}{3}x_3, 4x_4, \frac{1}{5}x_5, \ldots\right)$ for some $(x_1, x_2, x_3, \ldots, x_n, \ldots) \in D(T)$.

A small computation shows that $(x_1, x_2, x_3, \ldots, x_n, \ldots) = \left(1, \frac{1}{4}, 1, \frac{1}{16}, 1, \ldots\right)$ is not in H. Now let us calculate $\sigma_{\mathrm{p}}(T^*T)$. Since T^*T is one to one, we have $0 \notin \sigma_{\mathrm{p}}(T^*T)$. And $\sigma_{\mathrm{p}}(T^*T) = \left\{1, 4, \frac{1}{9}, 16, \frac{1}{25}, \ldots\right\}$. Since the spectrum is closed, $0 \in \sigma(T^*T)$, which is an accumulation point of the sequence $\left\{\frac{1}{n^2}: n \in \mathbb{N}\right\}$. Also

$$\sigma_{\mathrm{p}}(T) = \left\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, \ldots\right\} \subseteq \sigma(T).$$

In this case $\gamma(T) = 0 = d(T)$. This example also illustrates Theorems 3.3, 4.1, 4.3, 4.4 and 4.6.

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