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Integral Equations and Operator Theory

Continuity of the (n, ϵ)-Pseudospectrum in Banach Algebras

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Abstract. Let $\epsilon > 0$, n a non-negative integer, and A a complex unital Banach algebra. Define $\gamma_n : A \times \mathbb{C} \to [0, \infty]$ by

$$\gamma_n(a, z) = \begin{cases} \|(z-a)^{-2^n}\|^{-1/2^n}, & \text{if } (z-a) \text{ is invertible} \\ 0, & \text{if } (z-a) \text{ is not invertible.} \end{cases}$$

The (n, ϵ)-pseudospectrum $\Lambda_{n,\epsilon}(a)$ of an element $a \in A$ is defined by $\Lambda_{n,\epsilon}(a) := \{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) \leq \epsilon\}$. We show that γ_0 is Lipschitz on $A \times \mathbb{C}$, γ_n is uniformly continuous on bounded subsets of $A \times \mathbb{C}$ for $n \geq 1$, and γ_n is Lipschitz on some particular domains for $n \geq 1$. Using these properties, we establish that the map $(\epsilon, a) \mapsto \Lambda_{n,\epsilon}(a)$ is continuous at (ϵ_0, a_0) if and only if the level set $\{\lambda \in \mathbb{C} : \gamma_n(a_0, \lambda) = \epsilon_0\}$ does not contain any non-empty open set. In particular, this happens when a is a compact operator on a Banach space or a bounded operator on a Hilbert space or on an L^p space with $1 \leq p \leq \infty$. We also give examples of operators where this condition is not satisfied, and consequently, the map is not continuous.

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1. Introduction

The task of computing spectra of operators on Banach spaces, or more generally, of elements of a Banach algebra, is both interesting as well as important, due to the connection of this problem to differential and integral equations and quantum mechanics. The main difficulty in this is that the map that takes an element a to its spectrum $\sigma(a)$ is, in general, not continuous (see [6,13]). It was pointed out by Hansen in [13] that because of this discontinuity, while computing the spectra using computers, a small error in the input data may lead to unacceptable errors in the result. Hence he suggested the computation of different sets though the final aim may be to estimate the spectrum. These sets though different from the spectra should be good approximations to spectra and at the same time should depend continuously on the elements. Hansen [13] also suggested that (n,ϵ)-pseudospectra are excellent candidates for such sets as they satisfy both these requirements in case of operators on Hilbert spaces. We begin by recalling the definition of an (n,ϵ)-pseudospectrum.

Let A be a unital complex Banach algebra and $a \in A$. For an integer $n \geq 0$ and $\epsilon > 0$, the (n,ϵ)-pseudospectrum of a (see [12]) is defined by

$$\Lambda_{n,\epsilon}(a) = \{\lambda \in \mathbb{C} : \gamma_n(a,\lambda) \leq \epsilon\},$$

where $\gamma_n(a,.) : \mathbb{C} \to [0,\infty]$ is defined by

$$\gamma_n(a,z) = \begin{cases} \|(z-a)^{-2^n}\|^{-1/2^n}, & \text{if } (z-a) \text{ is invertible} \\ 0, & \text{if } (z-a) \text{ is not invertible.} \end{cases}$$

The $(0,\epsilon)$-pseudospectrum is popularly known as ϵ-pseudospectrum, denoted by $\Lambda_\epsilon(a)$, and has been studied extensively for non-normal matrices and operators over the last few decades. The monograph [23] by Trefethen and Embree contains a lot of information on the ϵ-pseudospectrum and its applications. Also refer to [2–5,9,16,18,21,22] for some pioneering work on ϵ-pseudospectra.

The notion of (n,ϵ)-pseudospectrum is relatively new, and this theory has emerged as a generalization of ϵ-pseudospectral theory. It was introduced originally for operators on separable Hilbert spaces by Hansen in [12] as a tool for the numerical approximation of spectral problems and was further developed for Banach space operators in [19] and to the elements of an arbitrary Banach algebra in [7,15].

One of the significant changes that appear while studying these sets in Banach space setting is that the continuity property may be lost. The possible occurrence of a constant resolvent norm of a Banach space operator on an open set is cited to be the main reason of discontinuity of the ϵ-pseudospectrum map. In this regard, we refer to the article [1], which gathers a good amount of information on the convergence of pseudospectra for Banach space and Hilbert space operators. Whereas the discontinuity of the ϵ-pseudospectrum map $a \mapsto \Lambda_\epsilon(a)$ is well known (for instance, see [17, Example 4.9] and [20]) for Banach space operators, no significant study has been done so far for (n,ϵ)-pseudopsectrum in a general Banach algebra setting.

In Sect. 3, we establish some essential properties of the approximating functions γ_n. We show that the functions γ_n are continuous for all $n \geq 0$, uniformly continuous on bounded sets (in fact, on any unbounded strip $\{(a,z) : \|a\| \leq R\}$ for any $R > 0$) for $n \geq 1$ and Lipschitz on some "particular" domains in $A \times \mathbb{C}$ for $n \geq 1$ (see Theorem 3.1). Further, we provide some counter examples to show that, in contrast to γ_0, the functions γ_n with $n \geq 1$ are neither Lipschitz on bounded sets, nor are uniformly continuous on $A \times \mathbb{C}$ (see Examples 3.2, 3.3).

In Sect. 4, using some of these properties of the functions γ_n, we collect several equivalent conditions for the continuity of the map $(\epsilon, a) \mapsto \Lambda_{n,\epsilon}(a)$ at (ϵ_0, a_0). Not surprisingly, these are equivalent to the condition that the level set $\{\lambda \in \mathbb{C} : \gamma_n(a_0,\lambda) = \epsilon_0\}$ does not contain any non-empty open subset.

The question on the above level set to contain a non-empty open set goes back to Globevnik (see [11]), who was the first to pose the question "can $\|(\lambda - a)^{-1}\|$ be constant on an open subset of the resolvent set $\rho(a)$ of a?" and answered it partially by showing that the above was not possible when (i) $\rho(a)$ is connected, and (ii) a is a bounded linear operator on a complex uniformly convex Banach space (see Definition 2.1). Independently, some related results were again established by Böttcher and Daniluk (see [2,3]). Shargorodosky combined their results and proved that the resolvent norm of a bounded linear operator can not be constant on an open set when the underlying space X or its dual X' is a complex uniformly convex Banach space (see [20]). This result covers finite dimensional spaces, Hilbert spaces and $L^p(\mu)$ spaces with $1 \leq p \leq \infty$. Further, he showed the existence of a bounded linear operator A on a Banach space X with $\|(A - \lambda)^{-1}\|$ being constant in a neighborhood of $\lambda = 0$ (see [20, Theorem 3.1]).

Given any $m \geq 1$, we establish the existence of a bounded invertible linear operator T on a Banach space X such that $\|(T - z)^{-m}\|$ is constant in an open set (see Example 4.10), and hence the pseudospectral map is discontinuous.

The essential spectra of bounded linear operators T are the (usual) spectra of the respective coset $T + K(X)$ in the Calkin algebra $B(X)/K(X)$. Thus, trying to use pseudospectral and n-pseudospectral techniques for the approximation of essential spectra naturally raises the necessity to have these tools available in Banach algebras.

Therefore the aim of the present paper is twofold: extending previous results on ϵ-pseudospectra in the operator context to (n, ϵ)-pseudospectra, and developing these tools in the more general Banach algebra case.

2. Notation, Basic Definitions and Known Results

Throughout, by a Banach algebra we mean a complex Banach algebra with unity 1 and $\|1\| = 1$. For $a \in A$, the spectrum $\sigma(a)$ of a is defined by

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible}\}.$$

The complement of $\sigma(a)$ is known as the resolvent set of a, and it will be denoted by $\rho(a)$. $B(X)$ and $K(X)$ denote the set of all bounded linear operators and compact operators on a complex Banach space X respectively. For $r > 0$ and $\lambda \in \mathbb{C}$, $B(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| < r\}$, $D(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| \leq r\}$. For $\Omega \subseteq \mathbb{C}$ and $\delta > 0$, $\Omega + D(0, \delta) := \underset{\lambda \in \Omega}{\cup} D(\lambda, \delta)$. $\mathrm{cl}(\Omega)$ will denote the closure of Ω. We will discuss the convergence of a sequence of non-empty compact sets in the complex plane with respect to the Hausdorff metric, defined by the following: for two non-empty compact sets $\Omega_1, \Omega_2 \subseteq \mathbb{C}$,

$$d_H(\Omega_1, \Omega_2) = \max\{\sup_{\lambda \in \Omega_2} d(\lambda, \Omega_1), \sup_{\lambda \in \Omega_1} d(\lambda, \Omega_2)\},$$

where $d(\lambda, \Omega) = \inf_{\mu \in \Omega} |\lambda - \mu|$.

Definition 2.1 (*Complex uniformly convex* [10,20]). A Banach space X is said to be complex uniformly convex (uniformly convex) if $\forall\, \epsilon > 0\, \exists\, \delta > 0$ such that

$$(x, y \in X,\ \xi \in D(0,1)(|\xi| \leq 1, \xi \in \mathbb{R}),\ \|x + \xi y\| \leq 1 \ \text{ and } \|y\| \geq \epsilon) \\ \Rightarrow \|x\| \leq 1 - \delta.$$

Remark 2.2. Since $\|\bar{\xi}x + \xi\bar{\xi}y\| = |\bar{\xi}|\|x + \xi y\|\ \forall \xi \in \mathbb{C}$, it follows that uniform convexity implies complex uniform convexity. So Hilbert spaces, ℓ^p and L^p spaces with $1 < p < \infty$ are complex uniformly convex. It is known that L^1 is complex uniformly convex (see [10]) but not uniformly convex, L^∞ is not complex uniformly convex, but its dual $(L^\infty)'$ is complex uniformly convex (see [20]).

Definition 2.3 (*Analytic map*). Let Ω be an open subset of $\mathbb{C}$. A function $f : \Omega \to A$ is said to be analytic on Ω if for every $\lambda_0 \in \Omega$ there is an element of A, denoted by $f'(\lambda_0)$, such that

$$\lim_{\lambda \to \lambda_0} \left\| \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0) \right\| = 0.$$

Theorem 2.4 (Maximum Modulus Theorem [8, p. 230]). *Let Ω be an open connected subset of $\mathbb{C}$ and X be a Banach space. Suppose $f : \Omega \to X$ is an analytic map. Then $\|f(z)\|$ has no maximum on Ω, unless $\|f(z)\|$ is identically constant.*

3. Properties of the Functions γ_n

In this section, we explore several useful properties of the functions γ_n which play a vital role in the computation of (n, ϵ)-pseudospectra. Whereas some elementary properties of γ_n have been studied in [7], a more detailed investigation is done in this section.

Theorem 3.1. *Let A be a Banach algebra. Let $n \geq 0$. For $(a, z) \in A \times \mathbb{C}$, $\|(a, z)\| := \|a\| + |z|$. Then the following statements hold.*

1. $|\gamma_0(a + b, z + w) - \gamma_0(a, z)| \leq \|b\| + |w|\ \forall\, (a, z), (b, w) \in A \times \mathbb{C}$. *Thus γ_0 is Lipschitz on $A \times \mathbb{C}$.*
2. *Let $m = 2^n, p(x) = \sum_{k=1}^{m} \binom{m}{k} x^{m-k}, (a, z), (b, w) \in A \times \mathbb{C}$ with $\|b\| + |w| \leq 1$. Then*

$$|\gamma_n(a + b, z + w) - \gamma_n(a, z)| \leq (p(\|a\| + |z|)^{1/m})(\|b\| + |w|)^{1/m}.$$

   Thus γ_n is continuous on $A \times \mathbb{C}$. Moreover, it is Hölder continuous and hence it is uniformly continuous on bounded subsets of $A \times \mathbb{C}$.
3. *$\gamma_n(a, z)$ is Lipschitz on domains where "z is dominating".*
   More precisely, let $n > 0, m = 2^n, R > 0$ and $0 < 1 - \frac{m^{1/m}}{2} < r < \frac{1}{2}$. Let $A(R, r) := \{(a, z) \in A \times \mathbb{C} : \|a\| \leq R, |z| \geq \frac{R+1}{r}\}$. Then $\forall (a, z) \in A(R, r)$ and $(b, w) \in A \times \mathbb{C}$ with $\|b\| + |w| \leq 1$, we have

$$|\gamma_n(a+b,z+w)-\gamma_n(a,z)| \le \frac{(1+r)^{m+2}}{(1-r)^{4m}}(\|b\|+|w|).$$

Thus γ_n is Lipschitz on $A(R,r)$.

4. *γ_n is uniformly continuous on $\{(a,z)\in A\times\mathbb{C} : \|a\|\le R\}$ for each $R>0$.*

Proof. 1. Let $a\in A$. Define

$$\gamma(a) := \begin{cases} \|a^{-1}\|^{-1}, & \text{if } a \text{ is invertible} \\ 0, & \text{otherwise.}\end{cases}$$

Note that for every $a\in A$,

$$\|a\| = \sup\left\{\frac{\|ax\|}{\|x\|} : x\in A, x\neq 0\right\} = \sup\{\|ax\| : x\in A, \|x\|=1\}.$$

Then for every invertible a

$$\gamma(a) = \|a^{-1}\|^{-1} = \inf\{\|ax\| : x\in A, \|x\|=1\}.$$

If $\|b\|<\|a^{-1}\|^{-1}$ then $a+b=a(1+a^{-1}b)$ is invertible since $\|a^{-1}b\|<1$. Thus, if a is invertible but $a+b$ is not invertible then necessarily $\|b\|\ge\|a^{-1}\|^{-1}$, hence $|\gamma(a)-\gamma(a+b)|\le\|b\|$. The same applies to the similar case with a not invertible but $a+b$ invertible. In case both are not invertible, $\gamma(a)=\gamma(a+b)=0$. If both are invertible then

$$\begin{aligned}\|(a+b)^{-1}\|^{-1} &= \inf\{\|(a+b)x\| : \|x\|=1\}\\ &\le \inf\{\|ax\|+\|bx\| : \|x\|=1\}\\ &\le \inf\{\|ax\| : \|x\|=1\}+\|b\| = \|a^{-1}\|^{-1}+\|b\|.\end{aligned}$$

Similarly, with a replaced by $a+b$ and b by $-b$, we finally get in all cases that $|\gamma(a)-\gamma(a+b)|\le\|b\|$. Since $\gamma_0(a,z)=\gamma(z-a)$ for all $a\in A$ and $z\in\mathbb{C}$, it follows that γ_0 is Lipschitz on $A\times\mathbb{C}$ with Lipschitz constant 1.

2. Let $a,b\in A$ with $\|b\|\le 1$.
For a and $a+b$ both invertible, we have

$$\begin{aligned}\|(a+b)^{-m}\|^{-\frac{1}{m}} &= (\inf\{\|(a+b)^m x\| : \|x\|=1\})^{\frac{1}{m}}\\ &\le (\inf\{\|a^m x\|+p(\|a\|)\|b\| : \|x\|=1\})^{\frac{1}{m}}\\ &\le (\inf\{\|a^m x\| : \|x\|=1\}+p(\|a\|)\|b\|)^{\frac{1}{m}}\\ &\le (\inf\{\|a^m x\| : \|x\|=1\})^{1/m}+(p(\|a\|)\|b\|)^{\frac{1}{m}}\\ &= \|a^{-m}\|^{-\frac{1}{m}}+(p(\|a\|)\|b\|)^{\frac{1}{m}}\end{aligned}$$

and further,

$$\begin{aligned}\|(a+b)^{-m}\|^{-\frac{1}{m}} &\ge (\inf\{\|a^m x\|-p(\|a\|)\|b\| : \|x\|=1\})^{\frac{1}{m}}\\ &\ge (\inf\{\|a^m x\| : \|x\|=1\})^{1/m}-(p(\|a\|)\|b\|)^{\frac{1}{m}}\\ &\ge \|a^{-m}\|^{-\frac{1}{m}}-(p(\|a\|)\|b\|)^{\frac{1}{m}}.\end{aligned}$$

So, combining the above two, we have

$$|\|(a+b)^{-m}\|^{-\frac{1}{m}}-\|a^{-m}\|^{-\frac{1}{m}}| \le (p(\|a\|)\|b\|)^{\frac{1}{m}}.$$

Now suppose that a is invertible, but $a+b$ is not invertible. Then necessarily $\|b\| \geq \|a^{-1}\|^{-1}$ holds, and with the convention $\|(a+b)^{-1}\| = \infty$, we have $\|(a+b)^{-m}\|^{-1/m} = 0$. Therefore

$$\begin{aligned}\|a^{-m}\|^{-\frac{1}{m}} \leq \|a^{-1}\|^{-1} \leq \|b\| \leq \|b\|^{\frac{1}{m}} \\ \leq \|(a+b)^{-m}\|^{-\frac{1}{m}} + (p(\|a\|)\|b\|)^{\frac{1}{m}},\end{aligned}$$

as $\|b\| \leq 1$ and $p(\|a\|) \geq 1$ by the definition of p. Analogously, if $a+b$ is invertible, but a is not, then

$$\|(a+b)^{-m}\|^{-\frac{1}{m}} \leq \|(a+b)^{-1}\|^{-1} \leq \|a^{-m}\|^{-\frac{1}{m}} + (p(\|a\|)\|b\|)^{\frac{1}{m}}.$$

We therefore have

$$\left|\|(a+b)^{-m}\|^{-\frac{1}{m}} - \|a^{-m}\|^{-\frac{1}{m}}\right| \leq (p(\|a\|)\|b\|)^{\frac{1}{m}}$$

in all cases. Replacing a by $z-a$ and b by $w-b$ finally gives the claim.

3. Consider the function $f(x) = x^{-\frac{1}{m}}$. For sufficiently small h we have that (e.g., look at its Taylor expansion)

$$|f(x) - f(x \pm h)| \leq \frac{(1+r)|h|}{mx^{1+\frac{1}{m}}} \text{ uniformly for all } x \in \left[\frac{1}{2^m}, 2^m\right].$$

Next, observe that for all $\|a\| \leq R$, $\|b\| \leq 1$ and $|z| \geq \frac{1}{r}(R+1) \geq \frac{1}{r}(\|a\| + \|b\|)$,

$$\begin{aligned}|h| &:= \left|\left\|\left(1 - \frac{a+b}{z}\right)^{-m}\right\| - \left\|\left(1 - \frac{a}{z}\right)^{-m}\right\|\right| \\ &\leq \left\|\left(1 - \frac{a+b}{z}\right)^{-m} - \left(1 - \frac{a}{z}\right)^{-m}\right\| \\ &\leq \left\|\left(1 - \frac{a+b}{z}\right)^{-m}\right\| \left\|\left(1 - \frac{a}{z}\right)^{m} - \left(1 - \frac{a+b}{z}\right)^{m}\right\| \left\|\left(1 - \frac{a}{z}\right)^{-m}\right\| \\ &\leq \left(\frac{1}{1-r}\right)^{m} \sum_{k=1}^{m} \binom{m}{k} \left\|1 - \frac{a}{z}\right\|^{m-k} \left\|\frac{b}{z}\right\|^{k} \left(\frac{1}{1-r}\right)^{m} \\ &\leq \left(\frac{1}{1-r}\right)^{2m} \frac{\|b\|}{|z|} \left(m \left\|1 - \frac{a}{z}\right\|^{m-1}\right. \\ &\quad \left. + \frac{\|b\|}{|z|} \sum_{k=2}^{m} \binom{m}{k} \left(\frac{1}{1-r}\right)^{m-k} \left\|\frac{b}{z}\right\|^{k-2}\right).\end{aligned}$$

If $\frac{\|b\|}{|z|}$ is sufficiently small the last factor can actually be further estimated such that

$$|h| \leq \left(\frac{1}{1-r}\right)^{4m} m \frac{\|b\|}{|z|},$$

thus, with small $\|b\|$ this $|h|$ gets as small as desired, even uniformly for all large $|z|$. The last estimation can be done in the following way.

Suppose $K = \left(m \left\| 1 - \frac{a}{z} \right\|^{m-1} + \frac{\|b\|}{|z|} \sum_{k=2}^{m} \binom{m}{k} \left(\frac{1}{1-r}\right)^{m-k} \left\| \frac{b}{z} \right\|^{k-2} \right)$. Now it is enough to show that $K \le m \left(\frac{1}{1-r}\right)^{2m}$. Since $\frac{\|b\|}{|z|} < r < 1+r < \frac{1}{1-r}$, we have

$$
\begin{aligned}
K &\le m \left(\frac{1}{1-r}\right)^{m-1} + \left(\frac{1}{1-r}\right) \sum_{k=2}^{m} \binom{m}{k} \left(\frac{1}{1-r}\right)^{m-k+k-2} \\
&\le \left(\frac{1}{1-r}\right)^{m-1} \left(m + 2^m - \binom{m}{1} - \binom{m}{0} \right) \\
&\le 2^m \left(\frac{1}{1-r}\right)^{m-1} \\
&\le m \left(\frac{1}{1-r}\right)^{2m} \quad \left(\text{since } 2 < \frac{m^{1/m}}{(1-r)} \text{ and } \frac{1}{1-r} > 1\right),
\end{aligned}
$$

as desired. Now, we see for all $\|a\| \le R$ and $|z| \ge \frac{1}{r}(R+1)$ and small $\|b\|$ (independent of a and z) that

$$
\begin{aligned}
&\gamma_n(a,z) - \gamma_n(a+b,z)| \\
&\quad = \left| |z| \left\| \left(1 - \frac{a}{z}\right)^{-m} \right\|^{-\frac{1}{m}} - |z| \left\| \left(1 - \frac{a+b}{z}\right)^{-m} \right\|^{-\frac{1}{m}} \right| \\
&\quad = |z| \left| f\left(\left\| \left(1 - \frac{a}{z}\right)^{-m} \right\| \right) - f\left(\left\| \left(1 - \frac{a}{z}\right)^{-m} \right\| + h \right) \right|
\end{aligned}
$$

and further, by the above estimates for $f(x)$ and h,

$$
\begin{aligned}
|\gamma_n(a,z) - \gamma_n(a+b,z)| &\le |z| \frac{(1+r)}{m \|(1-\frac{a}{z})^{-m}\|^{1+\frac{1}{m}}} |h| \\
&\le |z| \frac{(1+r)\|(1-\frac{a}{z})^{m}\|^{1+\frac{1}{m}}}{m} \frac{m\|b\|}{(1-r)^{4m}|z|} \\
&= \frac{(1+r)^{m+2}}{(1-r)^{4m}} \|b\|.
\end{aligned}
$$

For perturbations $z+w$ of the second variable z take into account that $\gamma_n(a+b, z+w) = \gamma_n(a+b-w, z)$. Consequently, γ_n is Lipschitz on $\{(a,z) \in A \times \mathbb{C} : \|a\| \le R, |z| \ge \frac{1}{r}(R+1)\}$.

4. Let $R > 0$. By (3), it follows that γ_n is uniformly continuous on $\{(a,z) \in A \times \mathbb{C} : \|a\| \le R, |z| \ge \frac{1}{r}(R+1)\}$. Also by (2), γ_n is uniformly continuous on $\{(a,z) \in A \times \mathbb{C} : \|a\| \le R, |z| \le \frac{1}{r}(R+1)\}$. Hence γ_n is uniformly continuous on $\{(a,z) \in A \times \mathbb{C} : \|a\| \le R\}$.

□

We now give an example which shows that, in contrast to γ_0, the functions γ_n with $n \ge 1$ are not Lipschitz, even on bounded sets.

Example 3.2. Let $A(\delta) = a(\delta)V$ be the operator on $\ell^2(\mathbb{Z})$ given by the shift operator $V : (x_i) \mapsto (x_{i+1})$ and the operator of multiplication $a(\delta)I$ with the sequence $a(\delta) = (a(\delta)_i)$ where $a(\delta)_0 = \delta$ and $a(\delta)_i = 1$ for all $i \ne 0$.

Then $\|A(\delta)^m\| = 1$ and $\|A(\delta)^{-m}\|^{-1} = \delta\ \forall\, m \geq 1$. Let $n \geq 1$ and take $m = 2^n$. Then $\gamma_n(A(\delta), 0) = \delta^{\frac{1}{m}}$ for every $\delta \in (0,1)$. On the other hand, $B = bV$ with $b_0 = 0$ and $b_i = 1$ for all $i \neq 0$ is not invertible, hence $\gamma_n(B, 0) = 0$. Since $\|A(\delta) - B\| = \delta$ we have

$$\frac{\gamma_n(A(\delta), 0) - \gamma_n(B, 0)}{\|A(\delta) - B\|} \to \infty \text{ as } \delta \to 0.$$

Thus γ_n is not Lipschitz.

We finally give an example which shows that the functions γ_n with $n \geq 1$ are not uniformly continuous on $A \times \mathbb{C}$.

Example 3.3. Let $A(\delta) = a(\delta)V$ be the operator on $\ell^2(\mathbb{Z})$ given by the shift operator $V : (x_i) \mapsto (x_{i+1})$ and the operator of multiplication $a(\delta)I$ with the sequence $a(\delta) = (a(\delta)_i)$ where $a(\delta)_0 = \delta$ and $a(\delta)_i = \delta^{-1}$ for all $i \neq 0$ and $0 < \delta < 1$.

Then $\|A^m\| = \delta^{-m}$ and $\|A^{-m}\|^{-1} = \delta^{-m+2}\ \forall\, m > 1$. Let $n \geq 1$ and choose $m = 2^n$. Thus $\gamma_n(A(\delta), 0) = \delta^{-1+\frac{2}{m}} \geq 1$ for every $\delta \in (0,1)$. On the other hand, $B(\delta) = b(\delta)V$ with $b(\delta)_0 = 0$ and $b(\delta)_i = \delta^{-1}$ for all $i \neq 0$ is not invertible, hence $\gamma_n(B(\delta), 0) = 0$. Since $\|A(\delta) - B(\delta)\| = \delta \to 0$ as $\delta \to 0$ but

$$|\gamma_n(A(\delta), 0) - \gamma_n(B(\delta), 0)| \geq 1\ \forall\, \delta \in (0,1),$$

thus γ_n is not uniformly continuous.

The following proposition improves the result of Hansen [14] on the local uniform convergence of the approximating functions to their uniform convergence on the whole complex plane.

Proposition 3.4. *Let A be a Banach algebra, $a \in A$ and $n \geq 0$. Suppose $\{a_k\}$ is a sequence in A such that $a_k \to a$ as $k \to \infty$. Then $\gamma_n(a_k, z) \to \gamma_n(a, z)$ as $k \to \infty$ uniformly on $\mathbb{C}$.*

Proof. There exists $M > 0$ such that $\|a\| \leq M$ and $\|a_k\| \leq M\ \forall\, k$. Let $\eta > 0$. Since γ_n is uniformly continuous on $E \times \mathbb{C}$ where $E := \{b \in A : \|b\| \leq M\}$, by Theorem 3.1(4), $\exists\, \delta > 0$ such that

$$(b, z), (c, w) \in E \times \mathbb{C} \text{ and } \|b - c\| + |z - w| < \delta \Rightarrow |\gamma_n(b, z) - \gamma_n(c, w)| < \eta.$$

Again, there exists k_0 such that $\|a - a_k\| < \delta\ \forall\, k \geq k_0$. In particular, for all $k \geq k_0$, we have

$$|\gamma_n(a_k, z) - \gamma_n(a, z)| < \eta\ \forall z \in \mathbb{C}.$$

Thus $\{\gamma_n(a_k, z)\}$ converges uniformly with respect to $z \in \mathbb{C}$ to $\gamma_n(a, z)$. $\square$

4. Main Results

In [7], it is shown that the map $\epsilon \mapsto \Lambda_{n,\epsilon}(a)$ is right continuous. Thus the map is continuous whenever it is left continuous. In the following proposition, we study some equivalent conditions for the left discontinuity of the above map. These conditions are not 'completely new' in the sense that their analogies either for usual pseudospectra or for (n, ϵ)-pseudospectra for Hilbert space operators (see [14,17,20]) are known to be true. Here, we present those results in an organized way with elementary proofs in the generalized setting of a Banach algebra.

Proposition 4.1. *Let A be a Banach algebra, $n \geq 0$ and $a \in A$. Then for a fixed $\epsilon_0 > 0$, the following statements are equivalent:*

1. *The map $\epsilon \mapsto \Lambda_{n,\epsilon}(a)$ is left discontinuous at ϵ_0.*
2. *The level set $\{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) = \epsilon_0\}$ contains a non-empty open set.*
3. *$cl(\{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) < \epsilon_0\}) \subsetneq \{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) \leq \epsilon_0\}$.*

Proof. For $\epsilon > 0$, let $A_\epsilon = \Lambda_{n,\epsilon}(a)$.
$1 \Rightarrow 2$. Suppose 1 holds. Hence $\exists\, r > 0$ such that $\forall\, \delta > 0\ \exists\, \epsilon > 0$ such that

$$\epsilon_0 - \delta < \epsilon < \epsilon_0 \text{ and } d_H(A_\epsilon, A_{\epsilon_0}) \geq r.$$

Consider a sequence $\{\delta_m\}$ such that $0 < \delta_m < \epsilon_0\ \forall\, m$ and $\delta_m \to 0$ as $m \to \infty$. Then for each m, $\exists\, \epsilon_m > 0$ such that

$$\epsilon_0 - \delta_m < \epsilon_m < \epsilon_0 \text{ and } d_H(A_{\epsilon_m}, A_{\epsilon_0}) \geq r.$$

Now $A_{\epsilon_m} \subseteq A_{\epsilon_0}\ \forall\, m$. Hence for each m, $\exists\, \lambda_m \in A_{\epsilon_0}$ such that $d(\lambda_m, A_{\epsilon_m}) \geq r$. Since A_{ϵ_0} is compact, $\{\lambda_m\}$ has a convergence subsequence. By renaming λ_m, if required, we may assume that $\lambda_m \to \lambda_0 \in A_{\epsilon_0}$, as $m \to \infty$. Since $\lambda_0 \in A_{\epsilon_0}$, we have $\gamma_n(a, \lambda_0) \leq \epsilon_0$. On the other hand, since $\lambda_m \notin A_{\epsilon_m}$, we have $\gamma_n(a, \lambda_m) > \epsilon_m > \epsilon_0 - \delta_m$ for each m. Since $\gamma_n(a, z)$ is continuous at each $z \in \mathbb{C}$ (by Theorem 3.1(2)), taking limit as $m \to \infty$, we have $\gamma_n(a, \lambda_0) \geq \epsilon_0$. Hence $\gamma_n(a, \lambda_0) = \epsilon_0$.

Also $\exists\, m_0$ such that $|\lambda_m - \lambda_0| < \frac{r}{2}\ \forall\, m \geq m_0$. Hence, for $z \in B(\lambda_0, \frac{r}{2})$, we have

$$|\lambda_m - z| \leq |\lambda_m - \lambda_0| + |\lambda_0 - z| < \frac{r}{2} + \frac{r}{2} = r\ \forall\, m \geq m_0.$$

Since $d(\lambda_m, A_{\epsilon_m}) \geq r$, we get $z \notin A_{\epsilon_m}\ \forall\, m \geq m_0$. Hence

$$\gamma_n(a, z) > \epsilon_m > \epsilon_0 - \delta_m\ \forall\, m \geq m_0.$$

Hence $\gamma_n(a, z) \geq \epsilon_0$. Thus we have proved that $\|(\lambda_0 - a)^{-2^n}\| = \frac{1}{{\epsilon_0}^{2^n}}$ and $\|(z - a)^{-2^n}\| \leq \frac{1}{{\epsilon_0}^{2^n}}\ \forall\, z \in B(\lambda_0, \frac{r}{2})$. By Theorem 2.4, $\|(z - a)^{-2^n}\| = \frac{1}{{\epsilon_0}^{2^n}}$ $\forall\, z \in B(\lambda_0, \frac{r}{2})$. Thus $B(z_0, \frac{r}{2}) \subseteq \{z \in U : f(z) = \epsilon_0\}$.
$2 \Rightarrow 3$. Straightforward.
$3 \Rightarrow 1$. Assume that $\text{cl}\{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) < \epsilon_0\} \subsetneq \{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) \leq \epsilon_0\}$. Then $\exists\, z_0 \in \mathbb{C}$ and $r > 0$ such that $\gamma_n(a, z_0) = \epsilon_0$ and $\gamma_n(a, z) \geq \epsilon_0\ \forall\, z \in B(z_0, r)$. Thus $\|(z_0 - a)^{-2^n}\| = \frac{1}{{\epsilon_0}^{2^n}}$ and $\|(z - a)^{-2^n}\| \leq \frac{1}{{\epsilon_0}^{2^n}}\ \forall z \in B(z_0, r)$. By Theorem 2.4, we have $\|(z - a)^{2^n}\| = \frac{1}{{\epsilon_0}^{2^n}}\ \forall\, z \in B(z_0, r)$. Hence $\{z \in \mathbb{C} : \gamma_n(a, z) = \epsilon_0\}$ contains $B(z_0, r)$. Consequently, $d(z_0, \Lambda_{n,\epsilon}(a)) \geq r\ \forall\, \epsilon < \epsilon_0$, which yields left discontinuity at ϵ_0. $\square$

Remark 4.2. Now we would like to point out an error in the proof of $2 \Rightarrow 3$ part of Proposition 4.1 in [17]. The fact that λ_0 depends on ϵ and its value changes as $\epsilon \to \epsilon_0$ was ignored in that proof. This error is fixed in the proof of above proposition by considering λ_0 as a limit point of the sequence $\{\lambda_n\}$. This is a more general result as it includes $(0, \epsilon)$-pseudospectra.

Remark 4.3. To prove the continuity of the pseudospectrum $\Lambda_\epsilon(a)$ with respect to a, the authors of [17] used the well known inclusion $\Lambda_\epsilon(a+b) \subseteq \Lambda_{\epsilon+\|b\|}(a)\ \forall\, a,b \in A$. In fact, if $\|b\| < \epsilon$, then using Theorem 3.1(1), it is possible to prove

$$\Lambda_{\epsilon-\|b\|}(a) \subseteq \Lambda_\epsilon(a+b) \subseteq \Lambda_{\epsilon+\|b\|}(a).$$

However, the inclusion $\Lambda_{n,\epsilon}(a+b) \subseteq \Lambda_{n,\epsilon+\|b\|}(a)$ need not be true, in general. See Example 2.14 in [7] and choose $\epsilon = 0.1$. In the following lemma, we propose a possible modification which provides an estimation of (n,ϵ)-pseudospectra.

Lemma 4.4. *Let A be a Banach algebra, $n \geq 0$ and $a_0 \in A$. Then for every $\eta > 0$ there exists $\delta > 0$ such that whenever $0 < \eta \leq \epsilon$ and $\|a_0 - a\| < \delta$, it follows that*

$$\Lambda_{n,\epsilon-\eta}(a_0) \subseteq \Lambda_{n,\epsilon}(a) \subseteq \Lambda_{n,\epsilon+\eta}(a_0).$$

Proof. Let $\eta > 0$. Since γ_n is uniformly continuous on $E := \{(a,z) \in A \times \mathbb{C} : \|a\| < \|a_0\| + 1\}$ by Theorem 3.1(4), there exists $0 < \delta < 1$ such that for all $z \in \mathbb{C}$

$$|\gamma_n(a_0,z) - \gamma_n(a,z)| < \eta \text{ if } \|a_0 - a\| < \delta.$$

The required inclusion follows immediately. $\square$

Definition 4.5 (*G_n-classes* [7]). For $n \geq 1$, an element $a \in A$ is said to be of G_n-class if $\gamma_{n-1}(a,\lambda) = d(\lambda, \sigma(a))\ \forall \lambda \in \mathbb{C}$.

Remark 4.6. From the above definition, it is clear that a is of G_n-class if and only if $\Lambda_{n-1,\epsilon}(a) = \sigma(a) + D(0,\epsilon)\ \forall\, \epsilon > 0$ and the G_n-classes form an increasing sequence as n increases, i.e., $G_n \subseteq G_{n+1}$ for all n. In [7], an example is given to show that this inclusion can be proper. Since the normal elements in a C^*-algebra are of G_1-class, the treatment of G_n-classes will automatically include normal elements.

Remark 4.7. Lemma 4.4 provides us a way to approximate $\Lambda_{n,\epsilon}(a)$. For example, if a_0 is of G_{n+1}-class, then it follows that for every $\eta > 0$ there exists $\delta > 0$ such that whenever $0 < \eta \leq \epsilon$ and $\|a_0 - a\| < \delta$, then

$$\sigma(a_0) + D(0,\epsilon-\eta) \subseteq \Lambda_{n,\epsilon}(a) \subseteq \sigma(a_0) + D(0,\epsilon+\eta).$$

In particular, it is easy to see that for $1 \leq p \leq \infty$, a diagonal matrix $a_0 :=$diag $(\lambda_1,\ldots,\lambda_n)$, viewed as an operator on $X := (\mathbb{C}^n, \|\cdot\|_p)$ is of G_1-class. Hence for every $\eta > 0$ there exists $\delta > 0$ such that for $\epsilon \geq \eta$ and $a \in B(X)$, $\|a - a_0\| < \delta$, we have

$$\bigcup_{i=1}^{n} D(\lambda_i, \epsilon-\eta) \subseteq \Lambda_{n,\epsilon}(a) \subseteq \bigcup_{i=1}^{n} D(\lambda_i, \epsilon+\eta).$$

Now we have developed the necessary tools and are ready to prove our main result in the following theorem. This extends classical continuity results known for pseudospectra in the operator context to the generalized setting of (n,ϵ)-pseudospectra in the context of Banach algebras.

Theorem 4.8. *Let A be a Banach algebra, $n \geq 0$, $a_0 \in A$ and $\epsilon_0 \in \mathbb{R}_+$. Then the following statements are equivalent.*

1. *The map $\epsilon \mapsto \Lambda_{n,\epsilon}(a_0)$ is continuous at ϵ_0.Now we have developed the necessary tools and are ready to prove our*
2. *The map $a \mapsto \Lambda_{n,\epsilon_0}(a)$ is continuous at a_0.*
3. *The map $(\epsilon, a) \mapsto \Lambda_{n,\epsilon}(a)$ is continuous at (ϵ_0, a_0) with respect to the metric in the domain given by*
$$\|(\epsilon_1, a_1)-(\epsilon_2, a_2)\|=\|a_1-a_2\|+|\epsilon_1-\epsilon_2| \ \forall\, a_1, a_2 \in A \text{ and } \epsilon_1, \epsilon_2 \in \mathbb{R}_+,$$
and the Hausdorff metric in the codomain.
4. *The level set $\{\lambda \in \mathbb{C} : \gamma_n(a_0, \lambda) = \epsilon_0\}$ does not contain any non-empty open set.*
5. *$cl(\{\lambda \in \mathbb{C} : \gamma_n(a_0, \lambda) < \epsilon_0\}) = \{\lambda \in \mathbb{C} : \gamma_n(a_0, \lambda) \leq \epsilon_0\}$.*

Proof. Equivalence of 1, 4 and 5 follows from Proposition 4.1.

$1 \Rightarrow 3$. Let $0 < \eta < \frac{\epsilon_0}{2}$. It follows from Lemma 4.4 that $\exists\, \delta > 0$ with $\delta < \eta$ such that $\forall\, b \in A$ and $\epsilon > 0$ with $\|a_0 - b\| + |\epsilon - \epsilon_0| < \delta$, we have
$$d_H(\Lambda_{n,\epsilon}(b), \Lambda_{n,\epsilon_0}(a_0)) \leq d_H(\Lambda_{n,\epsilon_0-2\eta}(a_0), \Lambda_{n,\epsilon_0+2\eta}(a_0)) \to 0 \text{ as } \eta \to 0,$$
because of the hypothesis. Thus the map $(\epsilon, a) \mapsto \Lambda_{n,\epsilon}(a)$ is jointly continuous at (ϵ_0, a_0).

$3 \Rightarrow 2$. Obvious.

$2 \Rightarrow 4$. Suppose 4 is false. Then $\exists\, \delta > 0$ and $z_0 \in \mathbb{C}$ such that
$$\gamma_n(a_0, z) = \epsilon_0 \ \forall\, z \in B(z_0, \delta).$$
Then $\gamma_n(a_0 - z_0, z - z_0) = \epsilon_0 \ \forall\, z \in B(z_0, \delta)$ (by Proposition 2.5(5) of [7]). Let $b_0 = a_0 - z_0$ and $w = z - z_0$. Thus $\gamma_n(b_0, w) = \epsilon_0 \ \forall\, w \in B(0, \delta)$. Now, let $w \in B(0, \frac{\delta}{2})$ and $k \geq 2$. Then $\frac{k}{k-1} \leq 2$. Hence $\frac{k}{k-1}w \in B(0, \delta)$. Also, by Proposition 2.5(6) of [7],
$$\begin{aligned}\gamma_n\left(\left(1 - \frac{1}{k}\right) b_0, w\right) &= \frac{1}{1 - \frac{1}{k}}\gamma_n\left(b_0, \frac{w}{1 - \frac{1}{k}}\right)\\ &= \frac{k}{k-1}\gamma_n\left(b_0, \frac{k}{k-1}w\right)\\ &= \frac{k}{k-1}\epsilon_0 > \epsilon_0.\end{aligned}$$
Hence $w \notin \Lambda_{n,\epsilon_0}((1 - \frac{1}{k})b_0)$. Thus
$$B\left(0, \frac{\delta}{2}\right) \cap \Lambda_{n,\epsilon_0}\left(\left(1 - \frac{1}{k}\right) b_0\right) = \emptyset \ \forall\, k \geq 2.$$
Hence
$$d_H\left(\Lambda_{n,\epsilon_0}((1 - \frac{1}{k})b_0), \Lambda_{n,\epsilon_0}(b_0)\right) \geq \frac{\delta}{2} \ \forall\, k \geq 2.$$
Now, again by Proposition 2.5 of [7],
$$\Lambda_{n,\epsilon_0}\left(\left(1 - \frac{1}{k}\right) b_0 + z_0\right) = z_0 + \Lambda_{n,\epsilon_0}((1 - \frac{1}{k})b_0).$$

Hence $\Lambda_{n,\epsilon_0}\left(\left(1-\frac{1}{k}\right)b_0+z_0\right)\cap B(z_0,\frac{\delta}{2})=\emptyset$ and $B(z_0,\delta)\subseteq\Lambda_{n,\epsilon_0}(a_0)$. Thus

$$d_H\left(\Lambda_{n,\epsilon_0}\left(\left(1-\frac{1}{k}\right)b_0+z_0\right),\Lambda_{n,\epsilon_0}(a_0)\right)\geq\frac{\delta}{2}\ \forall\, k\geq 2$$

and $(1-\frac{1}{k})b_0+z_0\to a_0$. This shows that the map $a\mapsto\Lambda_{n,\epsilon_0}(a)$ is discontinuous at a_0, contradiction. □

Remark 4.9. We now describe some class of elements in some Banach algebras where one and hence all the the conditions in Theorem 4.8 are fulfilled.

1. Suppose a_0 is of G_{n+1}-class. Then the level set

$$\{z\in\mathbb{C}:\gamma_n(a_0,z)=\epsilon_0\}=\{z\in\mathbb{C}:d(z,\sigma(a_0))=\epsilon_0\},$$

   clearly does not contain any non-empty open set.
2. Assume that the resolvent set $\rho(a_0)$ of a_0 is a connected subset of $\mathbb{C}$. An obvious modification of the Globvenik's result ([10]) shows that $\|(\lambda-a_0)^{-m}\|$ can not be constant on an open subset of $\rho(a_0)$ for any $m\geq 1$. This recovers the well known fact that if $A=B(X)$ and $K\in B(X)$ is compact, then the (n,ϵ)-pseudospectrum map is continuous at K.
3. Suppose $A=B(X)$ where either X or its dual X' is complex uniformly convex. Then $\|(\lambda-a_0)^{-2^n}\|$ can not take constant values on open sets (see [1] and [20, Theorem 2.6 & Corollary 2.7]).

Example 4.10. Given any integer $m\geq 1$, we now construct an example of a bounded linear operator on a Banach space such that the norm of the mth power of the resolvent of the operator is constant in an open set in the resolvent set of the operator. In [19], Seidel suggested this possible modification of the fundamental example of Shargorodsky in the following way. Let $X=\ell^\infty(\mathbb{Z})$ with the norm defined by

$$\|x\|=\sum_{k=0}^{m-1}|x_k|+\sup\{|x_k|:k\in\mathbb{Z}\backslash\{0,...,m-1\}\}. \tag{4.1}$$

It is easy to see that the above norm is equivalent to the usual sup norm on X. With this renorming of $\ell^\infty(\mathbb{Z})$, we now construct our desired operator. In partuicular, we show that there is a bounded linear operator $T:X\to X$ and a $\delta>0$ such that $B(0,\delta)\subseteq\rho(T)$, and $\|(T-\lambda)^{-m}\|$ is constant on $B(0,\delta)$. The proof is involved and different from the existing classical case [20, Theorem 2.3].

Let $M>4$. Define an operator $T:X\to X$ by

$$(Tx)_k=\alpha_k x_{k+1},\ \text{where}\ \alpha_k=\begin{cases}\frac{1}{M}, & k\in\{0,...,m-1\}\\ 1, & \text{otherwise}\ .\end{cases} \tag{4.2}$$

Clearly T is invertible and $(T^{-1}x)_k=\beta_k x_{k-1}$, where

$$\beta_k=\begin{cases}M, & k\in\{1,...,m\}\\ 1, & \text{otherwise.}\end{cases}$$

Note that $\|T\| = 1 + \frac{1}{M}$ and $\|T^{-1}\| = M$. Also it follows that $(T^{-m}x)_k = \beta_k\beta_{k-1}...\beta_{k-m+1}x_{k-m}$, $k \in \mathbb{Z}$. It can also be verified that $\|T^{-m}\| = M^m$. Note that for $|z| < \frac{1}{M}$,

$$(T - z)^{-1} = T^{-1}(I - zT^{-1})^{-1} = T^{-1}(I + R_z),$$

where $R_z = \sum\limits_{i=1}^{\infty} z^i T^{-i}$ with

$$\|R_z\| \le \frac{\|zT^{-1}\|}{1 - \|zT^{-1}\|} \le \frac{M|z|}{1 - M|z|} \to 0 \text{ as} |z| \to 0.$$

Again,

$$(T - z)^{-m} = T^{-m}(I - zT^{-1})^{-m} = T^{-m}(I + R_z)^m = T^{-m} + \Delta_z$$

where

$$\Delta_z = T^{-m}\left(\binom{m}{1}R_z + \binom{m}{2}R_z^2 + ... + \binom{m}{m}R_z^m\right).$$

Hence

$$\begin{aligned}\|\Delta_z\| &\le \|T^{-m}\|\left(\binom{m}{1}\|R_z\| + \binom{m}{2}\|R_z\|^2 + ... + \binom{m}{m}\|R_z\|^m\right)\\ &\le M^m((1 + \|R_z\|)^m - 1).\end{aligned}$$

Note that $\|\Delta_z\| \to 0$ as $|z| \to 0$. Again we may observe that with respect to the standard basis of $\ell^\infty(\mathbb{Z})$, the operator T^{-m} is lower triangular with non-zero entries (consisting of 1's and powers of M) only in its $(-m)$th diagonal and R_z is lower triangular with zero diagonal. Thus Δ_z is lower triangular with zero ith diagonal for $i \ge -m$.

$$[T^{-m}] = \begin{pmatrix}
 & \ddots & & & & & & \\
\ddots & & & 0 & & & & \\
 & 1 & & & \boxed{0} & & & \\
 & & M & & & 0 & & \\
 & & & M^2 & & & \ddots & \\
 & & & & \ddots & & & \\
 & & & & M^m & & & \\
 & & & & & M^{m-1} & & \\
 & & & & & \ddots & & \\
 & & & & & & M & \\
 & & & & & & & 1 \\
 & & & & & & & & \ddots
\end{pmatrix}$$

Let $x \in X$ such that $\|x\| \le 1$. Then

$$\begin{aligned}\|(T - z)^{-m}x\| &= \|T^{-m}x + \Delta_z x\|\\ &= \sum_{i=0}^{m-1} |(T^{-m}x + \Delta_z x)_i| + \sup_{i \ne 0,\dots,m-1} |(T^{-m}x + \Delta_z x)_i|. \quad (4.3)\end{aligned}$$

Thus

$$\begin{aligned}
&\|(T-z)^{-m}x\| \\
&\quad \leq \sum_{i=0}^{m-1} |(T^{-m}x)_i| + \sum_{i=0}^{m-1} |(\Delta_z(x-x_0e_0))_i| \\
&\qquad + \sup_{i\neq 0,\ldots,m} \{(T^{-m}x)_m + (\Delta_z(x-x_0e_0))_m, |(T^{-m}x+\Delta_z x)_i|\} \\
&\quad \leq \sum_{i=0}^{m-1} M^i|x_{i-m}| + \sum_{i=0}^{m-1} |(\Delta_z(x-x_0e_0))_i| \\
&\qquad + \sup_{i\neq 0,\ldots,m} \{M^m|x_0| + |(\Delta_z(x-x_0e_0))_m|, M^{m-1}|x_{i-m}| + |(\Delta_z x)_i|\}.
\end{aligned}$$

Since $\|\Delta_z\| \to 0$ as $|z| \to 0$, $\exists\ 0<\delta<\frac{1}{M}$ such that $|z|<\delta \Rightarrow \|\Delta_z\| < \frac{1}{m}$. Thus for all $|z|<\delta$, we get

$$\begin{aligned}
&\|(T-z)^{-m}x\| \\
&\quad \leq \sum_{i=0}^{m-1} M^i(\|x\|-|x_0|) + m\cdot\frac{1}{m}(\|x\|-|x_0|) \\
&\qquad + \sup\{M^m|x_0| + \frac{1}{m}(\|x\|-|x_0|), M^{m-1}(\|x\|-|x_0|) + \frac{1}{m}\|x\|\} \\
&\quad \leq \sup\{(\sum_{i=0}^{m-1} M^i + 1 + \frac{1}{m})(\|x\|-|x_0|) \\
&\qquad + M^m|x_0|, (\sum_{i=0}^{m-1} M^i + 1 + M^{m-1} + \frac{1}{m})(\|x\|-|x_0|) + \frac{1}{m}|x_0|\} \\
&\quad \leq \sup\{M^m(\|x\|-|x_0|) + M^m|x_0|, M^m(\|x\|-|x_0|) + M^m|x_0|\} \\
&\quad = M^m\|x\|.
\end{aligned}$$

Note that in the above computation, we have used the following inequality for $M \geq 4$,

$$\begin{aligned}
1+M+\ldots+M^{m-1} &\leq \frac{1}{2}\left(\frac{1}{2}\right)^m M^m + \frac{1}{2}\left(\frac{1}{2}\right)^{m-1} M^m + \ldots + \frac{1}{2}\left(\frac{1}{2}\right)^1 M^m \\
&< \frac{1}{2}M^m.
\end{aligned}$$

Take $x=e_0$. Then $\|x\|=1$ and using 4.3, we have

$$\|(T-z)^{-m}e_0\| = 0 + \sup_{i\neq 0,\ldots,m} \{M^m\cdot 1, 0 + (\Delta_z e_0)_i\} = M^m$$

for all $|z|<\delta$. Hence $\|(T-z)^{-m}\| = M^m$ for all $z \in B(0,\delta)$.

By taking $m=2^n$, we see that the level set $\{\lambda \in \mathbb{C} : \gamma_n(T,\lambda) = \frac{1}{M}\}$ contains $B(0,\delta)$. Hence, by Theorem 4.8, the map $S \mapsto \Lambda_{n,\epsilon}(S)$, $S \in B(X)$, is discontinuous at $S=T$. Note that T does not belong to any of the classes mentioned in Remark 4.9.

We have shown above the existence of a positive δ such that $\|(T-\lambda)^{-m}\|$ is constant in the open disc with the center at 0 and radius δ. Now, we actually estimate a particular value of δ. Observe that $\|R_z\| \leq \frac{M|z|}{1-M|z|} \to 0$ as $|z| \to 0$. Suppose $|z| < \delta$. Then $\|R_z\| < \frac{M\delta}{1-M\delta}$. Recall that

$$\|\Delta_z\| \leq \|T^{-m}\| \left(\binom{m}{1} \|R_z\| + \binom{m}{2} \|R_z\|^2 + ... + \binom{m}{m} \|R_z\|^m \right).$$

Suppose we choose $\delta > 0$ such that $\frac{M\delta}{1-M\delta} < 1$, that is, $\delta < \frac{1}{2M}$. Then $\|R_z\|^j < \|R_z\| < \frac{M\delta}{1-M\delta}$ $\forall j$. Hence

$$\begin{aligned} \|\Delta_z\| &\leq M^m \|R_z\| \left(\binom{m}{1} + \binom{m}{2} + ... + \binom{m}{m} \right) \\ &< M^{m+1} 2^m \frac{\delta}{1 - M\delta}. \end{aligned}$$

In the proof above, we wanted to choose $\delta > 0$ such that $\|\Delta_z\| < \frac{1}{m}$. This can be accomplished if $\delta > 0$ is chosen such that

$$M^{m+1} 2^m \frac{\delta}{1 - M\delta} < \frac{1}{m}, \quad \text{that is, } 2^m M^{m+1} \delta < \frac{1}{m} - \frac{M}{m}\delta.$$

This gives $\delta < \frac{1}{(M+m2^m M^{m+1})} < \frac{1}{2M}$. Thus we should take $\delta < \frac{1}{M+m2^m M^{m+1}}$.

Remark 4.11. In the concluding Remark of [19], the author has commented that the phenomenon of discontinuity of the (n, ϵ)-pseudospectra can be controlled by taking large values of n. The following proposition explains the same idea.

Proposition 4.12. *Fix $a \in A$ and $\epsilon_0 > 0$. Then the following statements hold.*

1. *For every $\eta_1 > 0$ there exists n_1 such that $d_H(\Lambda_{n,\epsilon}(a), \sigma(a) + D(0, \epsilon)) < \eta_1$ for all $n \geq n_1$ and all $\epsilon \leq \epsilon_0$. More precisely, $\sigma(a) + D(0, \epsilon) \subset \Lambda_{n,\epsilon}(a) \subset \sigma(a) + D(0, \epsilon + \eta_1)$.*
2. *For every $\eta_2 > 0$ there exists n_2 such that $d_H(\Lambda_{n,\epsilon_1}(a), \Lambda_{n,\epsilon_2}(a)) < |\epsilon_1 - \epsilon_2| + \eta_2$ for all $n \geq n_2$ and all $\epsilon_1, \epsilon_2 \leq \epsilon_0$.*
3. *For every $0 < \eta_3 < \epsilon_0$ there exists n_3 such that for all $n \geq n_3$ there is a $\delta(n) > 0$ such that $d_H(\Lambda_{n,\epsilon_1}(a), \Lambda_{n,\epsilon_2}(b)) < |\epsilon_1 - \epsilon_2| + \eta_3$ for all $\epsilon_1, \epsilon_2 \in [\frac{\eta_3}{4}, \epsilon_0]$ and all $b \in A$ with $\|a - b\| < \delta(n)$.*

Proof. The proofs of 1 and 2 are easy and 3 follows by Lemma 4.4. □

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