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Decomposition of the (n, ϵ)-pseudospectrum of an element of a Banach algebra

Kousik Dhara[1] · **S. H. Kulkarni**[2]

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Abstract

Let A be a complex Banach algebra with unit. For an integer $n \geq 0$ and $\epsilon > 0$, the (n, ϵ)-pseudospectrum of $a \in A$ is defined by

$$\Lambda_{n,\epsilon}(A, a) := \left\{\lambda \in \mathbb{C} : (\lambda - a) \text{ is not invertible in } A \text{ or } \|(\lambda - a)^{-2^n}\|^{1/2^n} \geq \frac{1}{\epsilon}\right\}.$$

Let $p \in A$ be a nontrivial idempotent. Then $pAp = \{pbp : b \in A\}$ is a Banach subalgebra of A with unit p, known as a reduced Banach algebra. Suppose $ap = pa$. We study the relationship of $\Lambda_{n,\epsilon}(A, a)$ and $\Lambda_{n,\epsilon}(pAp, pa)$. We extend this by considering first a finite family, and then an at most countable family of idempotents satisfying some conditions. We establish that under suitable assumptions, the (n, ϵ)-pseudospectrum of a can be decomposed into the union of the (n, ϵ)-pseudospectra of some elements in reduced Banach algebras.

Keywords Banach algebra · Spectrum · Pseudospectrum · (n, ϵ)-Pseudospectrum

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✉ Kousik Dhara
kousik.dhara1@gmail.com

S. H. Kulkarni
shk@iitpkd.ac.in

1 Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore Centre, 8th Mile, Mysore Road, Bangalore 560059, India

2 Department of Mathematics, Indian Institute of Technology Palakkad, Palakkad, Kerala 678557, India

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1 Introduction

It is known that the spectrum of the direct sum of a finite number of operators on the direct sum of Hilbert spaces is the union of their spectra (Problem 98 of [9]). The situation is slightly different if one considers an infinite direct sum of operators, and this has been studied in [4]. These operators arising naturally in the study of quantum mechanics and quantum field theory have been investigated by many researchers in the literature (see [8,15]).

Further, one can ask similar questions replacing spectrum by another subset of the complex plane, namely the (n, ϵ)-pseudospectrum. The (n, ϵ)-pseudospectrum of a linear operator on a Hilbert space was first introduced and studied by Hansen in [10–12] and subsequently was further explored for Banach space operators by Seidel [20] and to the elements of a Banach algebra in [6,13]. The advantages of studying these sets are that in contrast to spectra, the (n, ϵ)-pseudospectra are less sensitive in perturbations and approximate spectra for arbitrarily large n (see [11,12]).

The present note aims to show that the (n, ϵ)-pseudospectrum of an element of a Banach algebra can be decomposed into a union of the (n, ϵ)-pseudospectra of some elements of certain reduced Banach subalgebras. These subalgebras are expected to be finite dimensional and hence the computation of these sets is expected to be easier.

Throughout, by a Banach algebra A, we always mean a complex Banach algebra with unit 1 and $\|1\| = 1$. Let $p \in A$ be an idempotent, that is, $p^2 = p$. We always assume that p is non-trivial, that is, $p \neq 0$ and $p \neq 1$. Then $pAp = \{pap : a \in A\}$ is a subalgebra of A with unit p. Now, if $pa_n p$ converges to $x \in A$, then by continuity, $pa_n p$ also converges to pxp and hence $x = pxp \in pAp$. Thus pAp is a Banach algebra, known as a reduced Banach algebra. We study the relationship between the (n, ϵ)-pseudospectrum of $a \in A$ and the (n, ϵ)-pseudospectrum of $pap \in pAp$. The usual pseudospectra case (i.e., $n = 0$) has been dealt with in details in [16,18].

In large parts of this paper, we concentrate on the particular case with p_i being idempotents with $\sum p_i = 1$, often under additional condition $\|p_i\| = 1$. Further, a is assumed to commute with all p_i, which figuratively speaking means, that p_i somehow decompose A into a kind of "orthogonal" sum, and further a is "diagonal" with respect to this decomposition. A good example of this situation is provided by block diagonal operators.

We organize the paper as follows. In Sect. 2, we review basic definitions and introduce some notation used in this paper. In Sect. 3, we consider $a \in A$ and a finite family of idempotents $p_1, \ldots, p_l$ in A satisfying $ap_i = p_i a$ for each i and $\sum_{i=1}^{l} p_i = 1$, and answer the following question: when can one write

$$\Lambda_{n,\epsilon}(A, a) = \bigcup_{i=1}^{l} \Lambda_{n,\epsilon}(p_i A p_i, p_i a) \,\forall \epsilon > 0? \tag{1}$$

We show that the equality in (1) occurs when A has the following property:

$$\text{If } y \in A \text{ such that } yp_i = p_i y \,\forall i, \text{ then } \|y\| = \max_{1 \le i \le l} \|p_i y\|.$$

The last equality always occurs when A is a C^*-algebra and the idempotents are self adjoint (see Proposition 2). Further, we show that if a is of G_{n+1}-class (see Definition 6) such as self-adjoint and normal elements in a C^*-algebra, then (1) is satisfied (see Proposition 3). In Sect. 4, we consider at most countable family of idempotents $\{p_j : j \in I\}$ and study the invertibility of a by studying the invertibility of the elements $p_j a$ in $p_j A p_j$, $j \in I$ under suitable assumptions. Subsequently, we investigate the behaviour of the spectrum of a and the (n, ϵ)-pseudospectrum of a in terms of their reduced components (see Theorem 4).

2 Background and notation

In this section, we recall some basic definitions and results. Let A be a Banach algebra and $a \in A$. We identify the scalar $\lambda \in \mathbb{C}$ with $\lambda 1 \in A$. $B(X)$ denotes the Banach algebra of all bounded linear operators on a Banach space X with the operator norm.

Definition 1 The spectrum of a is defined by

$$\sigma(A, a) := \{\lambda \in A : \lambda - a \text{ is not invertible in } A\}, \text{ and}$$

the spectral radius of a is defined by

$$r(A, a) := \max\{|\lambda| : \lambda \in \sigma(A, a)\}.$$

Definition 2 The numerical range of a (see [1]) is defined by

$$V(a) := \{g(a) : g \in A', \|g\| = 1 = g(1)\},$$

where A' is the dual of A. An element a is Hermitian if its numerical range $V(a)$ is contained in the real line.

Definition 3 For $\epsilon > 0$, the ϵ-pseudospectrum of a is defined by

$$\Lambda_\epsilon(A, a) = \sigma(A, a) \cup \left\{\lambda \notin \sigma(A, a) : \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}\right\}.$$

For a detailed treatment of the ϵ-pseudospectrum, one may refer to [17], the comprehensive monograph by Trefethen and Embree [22] and the references therein. Also, we refer to [2,3,14,21] for more information.

Definition 4 For an integer $n \geq 0$, define the function $\gamma_n(A, ., .) : A \times \mathbb{C} \to [0, \infty)$ by

$$\gamma_n(A, b, \lambda) = \begin{cases} \|(\lambda - b)^{-2^n}\|^{-1/2^n}, & \text{if } \lambda - b \text{ is invertible} \\ 0, & \text{otherwise.} \end{cases}$$

It is known that the functions γ_n are continuous for all $n \geq 0$, uniformly continuous on bounded sets for $n \geq 1$ and Lipschitz on some "particular" domains in $A \times \mathbb{C}$ for $n \geq 1$ (see [7, Theorem 3.1]).

Definition 5 For an integer $n \geq 0$ and $\epsilon > 0$, the (n, ϵ)-pseudospectrum of a is defined by (see [10])

$$\Lambda_{n,\epsilon}(A, a) := \{\lambda \in \mathbb{C} : \gamma_n(A, a, \lambda) \leq \epsilon\}.$$

Remark 1 The notion of the (n, ϵ)-pseudospectrum is an extension of the theory of ϵ-pseudospectrum. It is known that the (n, ϵ)-pseudospectrum of a is a non-empty compact subset of the complex plane and it approximates the closed ϵ-neighbourhood of the spectrum of a as n tends to infinity (see [11,20] and [6, Theorem 2.8]).

3 Finite number of idempotents

In this section, we consider a finite number of idempotents $p_1, \ldots, p_l$ satisfying certain conditions and study the relationship between $\Lambda_{n,\epsilon}(A, a)$ and the sets $\Lambda_{n,\epsilon}(p_i A p_i, p_i a)$ for $i = 1, \ldots, l$. We begin with a well-known lemma (see [9,17]). The proof is included for the sake of completeness.

Lemma 1 *Let A be a Banach algebra.*

1. *Let p be an idempotent in A and $a \in A$ such that $ap = pa$. Then $\sigma(pAp, pa) \subseteq \sigma(A, a)$.*
2. *Let $p_1, \ldots, p_l$ be idempotents in A such that $\sum_{i=1}^{l} p_i = 1$. Suppose $a \in A$ is such that $ap_i = p_i a$ for each i. Then*

$$\sigma(A, a) = \bigcup_{i=1}^{l} \sigma(p_i A p_i, p_i a).$$

Hence $r(A, a) = \max\limits_{1 \leq i \leq l} r(p_i A p_i, p_i a)$.

Proof 1. If $\lambda \notin \sigma(A, a)$, then there exists $b \in A$ such that $(\lambda - a)b = b(\lambda - a) = 1$. So $(\lambda p - ap)b = p = b(\lambda p - ap)$ and hence $\lambda \notin \sigma(pAp, pa)$.

2. The inclusion $\bigcup\limits_{i=1}^{l} \sigma(p_i A p_i, p_i a) \subseteq \sigma(A, a)$ follows from 1. For the reverse inclusion, let $\lambda \notin \sigma(p_i A p_i, p_i a)$ for all $i \in \{1, \ldots, l\}$. Then for each i, there is $b_i \in A$ such that

$$(\lambda p_i - p_i a) p_i b_i p_i = p_i b_i p_i (\lambda p_i - p_i a) = p_i.$$

Then we have

$$(\lambda - a)\left(\sum_{i=1}^{l} p_i b_i p_i\right) = \left(\sum_{i=1}^{l} p_i b_i p_i\right)(\lambda - a) = \sum_{i=1}^{l} p_i = 1.$$

Hence $\lambda \notin \sigma(A, a)$. ◻

Lemma 2 *Let A be a Banach algebra, $p \in A$ be an idempotent, $n \geq 0$ and $m = 2^n$. Let $a \in A$ be such that $ap = pa$. Then*

$$\gamma_n(A, a, z) \leq \|p\|^{1/m} \gamma_n(pAp, pa, z)\ \forall z \in \mathbb{C}, \text{ and consequently}$$
$$\Lambda_{n,\epsilon}(pAp, pa) \subseteq \Lambda_{n, \|p\|^{1/m}\epsilon}(A, a)\ \forall \epsilon > 0.$$

Proof First note that if a is invertible in A, then $pa^{-1} = a^{-1}p$ is the inverse of pa in pAp. Suppose $z \notin \sigma(A, a)$. Then $p(z-a)^{-m}$ is the inverse of $p(z-a)^m = (p(z-a))^m$ in pAp. Also $\|p(z-a)^{-m}\| \leq \|p\| \|(z-a)^{-m}\|$. Thus

$$\begin{aligned}\gamma_n(A, a, z) &= \|(z-a)^{-m}\|^{-1/m} \\ &\leq \|p\|^{1/m} \|p(z-a)^{-m}\|^{-1/m} \\ &= \|p\|^{1/m} \gamma_n(pAp, pa, z).\end{aligned}$$

If $(z-a)$ is not invertible in A, then $\gamma_n(A, a, z) = 0$ and hence the above inequality holds trivially. ◻

Proposition 1 *Let A be a Banach algebra, $n \geq 0$, $m = 2^n$ and $\epsilon > 0$. Let $p_1, \ldots, p_l$ be idempotents in A such that $\sum_{i=1}^{l} p_i = 1$. Further, let $a \in A$ such that $ap_i = p_i a$ for all $i \in \{1, \ldots, l\}$ and $K := \max\limits_{i=1,\ldots,l} \|p_i\|^{1/m}$. Then for all $z \in \mathbb{C}$*

$$\begin{aligned}&\gamma_n(A, a, z) \leq K \min_{1 \leq i \leq l} \gamma_n(p_i a p_i, p_i a, z) \text{ and} \\ &\min_{1 \leq i \leq l} \gamma_n(p_i a p_i, p_i a, z) \leq l^{1/m} \gamma_n(A, a, z).\end{aligned} \tag{2}$$

Consequently,

$$\Lambda_{n, \epsilon l^{-1/m}}(A, a) \subseteq \bigcup_{i=1}^{l} \Lambda_{n,\epsilon}(p_i A p_i, p_i a) \subseteq \Lambda_{n, K\epsilon}(A, a).$$

Proof The first of (2) follows directly from Lemma 2. For the second part, first assume that $(z-a)^m p_i$ is invertible in $p_i A p_i$ with inverse $p_i b_i p_i$ for all i, then $(z-a)^m$ is invertible and $(z-a)^{-m} = \sum_{i=1}^{l} p_i b_i p_i$. Thus

$$\|(z-a)^{-m}\| = \|\sum_{i=1}^{l} p_i b_i p_i\| \leq l \max_{1 \leq i \leq l} \|p_i b_i p_i\|.$$

Hence

$$\|(z-a)^{-m}\|^{-1/m} \geq l^{-1/m} \min_{1 \leq i \leq l} \|p_i b_i p_j\|^{-1/m}.$$

This yields

$$\min_{1\leq i\leq l}\gamma_n(p_iAp_i,\, p_ia,\, z) \leq l^{1/m}\gamma_n(A,\, a,\, z). \tag{3}$$

Now, if $(z-a)p_{i_0}$ is not invertible in $p_{i_0}Ap_{i_0}$ for some i_0, then

$$\gamma_n(p_{i_0}Ap_{i_0},\, p_{i_0}a,\, z) = 0 = \min_{1\leq i\leq l}\gamma_n(p_iAp_i,\, p_ia,\, z).$$

Hence the second part (2) holds for all $z \in \mathbb{C}$. Thus

$$\lambda \in \Lambda_{n,\epsilon l^{-1/m}}(A,a) \iff \gamma_n(A,a,\lambda) \leq \epsilon l^{-1/m} \Rightarrow \min_{1\leq i\leq l}\gamma_n(p_iAp_i,\, p_ia,\, \lambda) \leq \epsilon.$$

Consequently $\lambda \in \Lambda_{n,\epsilon}(p_{i_0}Ap_{i_0},\, p_{i_0}a)$ for some i_o. Thus $\Lambda_{n,\epsilon l^{-1/m}}(A,a) \subseteq \bigcup_{i=1}^{l}\Lambda_{n,\epsilon}(p_iAp_i,\, p_ia)$. Hence the required results follow. □

Remark 2 With the assumption that $\|p_i\| = 1$, for each i, in general, the inclusion $\bigcup_{i=1}^{l}\Lambda_{n,\epsilon}(p_iAp_i,\, p_ia) \subseteq \Lambda_{n,K\epsilon}(A,a)$ can be proper. See [17, Example 3.9] for $l=2$ and $n=0$.

So when can we write $\Lambda_{n,\epsilon}(A,a) = \bigcup_{i=1}^{l}\Lambda_{n,\epsilon}(p_iAp_i,\, p_ia)\,\forall\epsilon > 0$? In the following theorem, we give a sufficient condition under which the above equality occurs.

Theorem 1 *Let A be a Banach algebra. Let $p_1,\ldots,p_l$ be idempotents in A such that $\sum_{i=1}^{l}p_i = 1$. Suppose $a \in A$ such that $ap_i = p_ia$ for each i. Further, assume that A has the following property:*

$$\textit{if } y \in A \textit{ such that } yp_i = p_iy\,\forall i, \textit{ then } \|y\| = \max_{1\leq i\leq l}\|p_iy\|. \tag{4}$$

Then for $n \geq 0$,

$$\gamma_n(A,a,\lambda) = \min_{1\leq i\leq l}\gamma_n(p_iAp_i,\, p_ia,\, \lambda)\,\forall\lambda \in \mathbb{C}. \tag{5}$$

Consequently,

$$\bigcup_{i=1}^{l}\Lambda_{n,\epsilon}(p_iAp_i,\, p_ia) = \Lambda_{n,\epsilon}(A,a)\,\forall\epsilon > 0.$$

Proof Let $m = 2^n$. If $\lambda \in \sigma(A,a)$, then (5) follows from Lemma 1(2). Let $\lambda \notin \sigma(a)$ and let $x = (\lambda - a)^m$. For each $i \in \{1,\ldots,l\}$, $xp_i = p_ix$ and $p_ix^{-1} = x^{-1}p_i$. Then the hypothesis says that $\|x^{-1}\| = \max_{1\leq i\leq l}\|p_ix^{-1}\|$. Hence

$$\|(\lambda-a)^{-m}\| = \max_{1\leq i\leq l}\|p_i(\lambda-a)^{-m}\|.$$

This yields (5). $\square$

The next proposition presents a sufficient condition for which the condition (4) holds.

Proposition 2 *Let A be a C^*-algebra. Let $p_1, \dots, p_l$ be self adjoint idempotents in A such that $\sum_{i=1}^{l} p_i = 1$. Then the condition (4) holds.*

Proof Let $a \in A$ such that $ap_i = p_i a \; \forall i$. Then

$$\begin{aligned}\|a\|^2 = \|a^*a\| = r(A, a^*a)& \\ &= \max_{1\le j\le l} r(p_jAp_j, p_j(a^*a)) \;\; \text{(using Lemma 1)}\\ &= \max_{1\le j\le l} r(p_jAp_j, (p_ja)^*(p_ja))\\ &= \max_{1\le j\le l} \|(p_ja)^*(p_ja)\| \;\; (\text{since } p_jAp_j \text{ is a } C^*\text{-algebra})\\ &= \max_{1\le j\le l} \|p_ja\|^2.\end{aligned}$$

This immediately gives the required result. $\square$

Definition 6 (G_n*-classes* [6]) Suppose A is a Banach algebra. For $n \geq 1$, an element $a \in A$ is said to be of G_n-class if $\gamma_{n-1}(A, a, z) = d(z, \sigma(A, a)) \; \forall z \in \mathbb{C}$.

Remark 3 Note that a is of G_n-class if and only if $\Lambda_{n-1,\epsilon}(A, a) = \sigma(A, a) + D(0, \epsilon)$ $\forall \epsilon > 0$. Further, $G_n \subseteq G_{n+1} \; \forall n$. It is known that the normal operators on a Hilbert space are of G_1-class (see [19]). In [6], an example of a bounded linear operator on $\ell^2(\mathbb{Z})$ is given to show the existence of non-normal elements in G_n-classes for some $n > 1$.

Proposition 3 *Let A be a Banach algebra and $\epsilon > 0$. Let $p_1, \dots, p_l$ be idempotents in A such that $\sum_{i=1}^{l} p_i = 1$. Let $a \in A$ be such that $ap_i = p_i a$ and $\|p_i\| = 1 \; \forall i \in \{1, \dots, l\}$. Further assume that a is of G_{n+1}-class for some n. Then*

$$\bigcup_{i=1}^{l} \Lambda_{n,\epsilon}(p_iAp_i, p_ia) = \Lambda_{n,\epsilon}(A, a).$$

Proof Let $\lambda \in \mathbb{C}$. Since a is of G_{n+1}-class, $\gamma_n(A, a, \lambda) = d(\lambda, \sigma(A, a))$. By Lemma 1, it follows that

$$\begin{aligned} d(\lambda, \sigma(A, a)) &= d(\lambda, \bigcup_{i=1}^{l} \sigma(p_iAp_i, p_ia))\\ &= \min_{1\le i\le l} d(\lambda, \sigma(p_iAp_i, p_ia))\end{aligned}$$

The required result is immediate by Proposition 2.5(3) of [6] and Proposition 1 with $K = 1$. $\square$

The following proposition is a generalization of a result on usual pseudospectra. See [16, Theorem 3.18].

Theorem 2 *Let A be a Banach algebra and $p_1, p_2, \dots, p_l$ be idempotents in A such that $p_j p_k = 0$ if $j \neq k$. Suppose there exists a function $g : \mathbb{R}_+^l \to \mathbb{R}$ such that*

$$\|x\| = g(\|p_1 x\|, \dots, \|p_l x\|) \ \forall\, x \in A \text{ satisfying } xp_i = p_i x \ \forall\, i.$$

Then the following statements hold.

1. $\|x\| = \max\limits_{1 \le i \le l} \|p_i x\| \ \ \forall\, x \in A$ *satisfying* $xp_i = p_i x \ \forall i$.
2. $\|p_j\| = 1 \ \forall\, j$.
3. *For each* j, p_j *is Hermitian.*
4. $\sum_{i=1}^{l} p_i = 1$.
5. *Further, if* $ap_i = p_i a$ *for all* i, *then*

$$\Lambda_{n,\epsilon}(A, a) = \bigcup_{i=1}^{l} \Lambda_{n,\epsilon}(p_i A p_i, p_i a).$$

Proof 1. It is proved in [16, Theorem 3.18] that the above hypotheses imply that

$$\|x\| = \max_{1 \le i \le l} \|p_i x\| \ \ \forall\, x \in A \text{ satisfying } xp_i = p_i x \ \forall i.$$

2. Follows by taking $x = 1$ in statement 1.
3. Let $j \in \{1, \dots, l\}$. In view of [1, Corollary 1.10.13], it is enough to show that

$$\|e^{itp_j}\| = 1 \ \forall t \in \mathbb{R}.$$

   Let $x = e^{itp_j} = \sum_{k=0}^{\infty} \dfrac{(itp_j)^k}{k!}$. Then $xp_j = p_j + itp_j + \frac{(itp_j)^2}{2} + \cdots = e^{it} p_j = p_j x$. Then $\|p_j x\| = |e^{it}| \|p_j\| = 1$. For $k \neq j$, $p_k x = p_k = xp_k$ since $p_k p_j = 0$. Hence $\|p_k x\| = \|p_k\| = 1 \ \forall\, k \neq j$, and so $\|x\| = 1$ by 1.
4. Let $x = 1 - (p_1 + \cdots + p_l)$. Then $p_j x = 0 \ \forall\, j$. Applying 1, we get $x = 0$.
5. Finally, we see that all the hypotheses of Theorem 1 are satisfied and hence 5 follows.

□

4 At most countable family of idempotents

In this section, we turn our attention to a family of idempotents $\{p_j : j \in I\}$, where I is an at most countable set. Note that the sum $\sum_{j \in I} p_j$ does not converge in the norm topology. So we may need additional hypotheses to obtain the earlier results.

Theorem 3 *Let A be a Banach algebra and I be an at most countable set. Let $\{p_j : j \in I\}$ be a family of idempotents in A satisfying $p_i p_j = 0$ for $i \neq j$, and*

$$\|x\| = \sup_{j\in I} \|p_j x\| \textit{ for all } x \in A \textit{ such that } p_j x = x p_j \; \forall\, j \in I. \tag{6}$$

Then the following statements hold.

1. $\|p_j\| = 1$ *for all* j.
2. p_j *is Hermitian for each* j.
3. *If* I *is a finite set, say* $I = \{1, \ldots, l\}$*, then* $1 = p_1 + \cdots + p_l$.
4. *Suppose* $a \in A$ *is invertible with inverse* a^{-1} *and* $ap_j = p_j a \; \forall\, j \in I$*. Then* $a^{-1}p_j = p_j a^{-1}$, $p_j a^{-1}$ *is the inverse of* $p_j a p_j$ *in* $p_j A p_j$ *for each* j *and* $\|a^{-1}\| = \sup_{j\in I}\|p_j a^{-1}\|$.
5. $\overline{\bigcup_{j\in I} \sigma(p_j A p_j, p_j a)} \subseteq \sigma(A, a)$.

Proof 1, 2 and 3 follow in a similar way as in Theorem 2. To see 4, note that $a^{-1}ap_j a^{-1} = a^{-1}p_j a a^{-1}$ and hence $p_j a^{-1} = a^{-1}p_j \; \forall\, j \in I$. Thus by (6), it follows that $\|a^{-1}\| = \sup_{j\in I}\|p_j a^{-1}\|$. Also, $p_j a^{-1} = p_j a^{-1} p_j \in p_j A p_j$ and $p_j a p_j p_j a^{-1} = p_j$. Similarly, $p_j a^{-1} p_j a p_j = p_j$. Hence $p_j a^{-1}$ is the inverse of $p_j a p_j$ in $p_j A p_j$. The last assertion is immediate. □

In the following theorem, we give a sufficient condition for the invertibility of an element $a \in A$ in terms of the invertibility of the elements $p_i a p_i \in p_i A p_i$ and we discuss its impact on the computation of the spectra and the (n, ϵ)-pseudospectra.

Theorem 4 *Let* A *be a Banach algebra. Suppose* $\{p_j : j \in I\}$ *is an at most family of idempotents in* A *satisfying* (6) *and further the family has the following property:*

$$\Big(a \in A,\; p_j a = a p_j \;\forall\, j \in I,\, p_j a \textit{ is invertible in } p_j A p_j \textit{ with inverse } p_j b_j p_j \textit{ for each } j \in I \textit{ and } \sup_{j\in I} \|p_j b_j p_j\| < \infty\Big) \Rightarrow a \textit{ is invertible in } A \textit{ and } \|a^{-1}\| = \sup_{j\in I}\|p_j b_j p_j\|. \tag{7}$$

Let $a \in A$ *such that* $ap_j = p_j a \;\forall j \in I$*. Then the following statements hold.*

1. $$\sigma(A, a) = \bigcup_{j\in I} \sigma(p_j A p_j, p_j a) \cup \{\lambda \in \mathbb{C} \setminus \bigcup_{j\in I} \sigma(p_j A p_j, p_j a) : \sup_{j\in I}\|p_j c_j p_j\| = \infty\}, \tag{8}$$

   where $p_j c_j p_j$ *is the inverse of* $p_j(\lambda - a)$ *in* $p_j A p_j$.
2. *For a non negative integer* n *and* $z \in \mathbb{C}$,

$$\gamma_n(A, a, z) = \inf_{j\in I} \gamma_n(p_j A p_j, p_j a, z).$$

3. *For* $\epsilon > 0$,

$$\Lambda_{n,\epsilon}(A,a) = \bigcup_{j\in I} \Lambda_{n,\epsilon}(p_jAp_j, p_ja) \cup \{\lambda \in \mathbb{C} : \inf_{j\in I} \gamma_n(p_jAp_j, p_ja, \lambda) = \epsilon\}.$$

Proof 1. By Theorem 3(4), $\bigcup_{j\in I} \sigma(p_jAp_j, p_ja) \subseteq \sigma(A,a)$. Let $\lambda \notin \sigma(A,a)$. Replacing a by $(\lambda - a)$ in Theorem 3(4), we have one side of the inclusion in (8). The other side of the inclusion follows from (7).

2. Suppose $z \notin \sigma(A,a)$. Then

$$\|(z-a)^{-2^n}\| = \sup_{j\in I} \|p_j(z-a)^{-2^n}\| \quad \text{(by hypothesis)},$$

and hence $\gamma_n(A,a,z) = \inf_{j\in I} \gamma_n(p_jAp_j, p_ja, \lambda)$.

Next suppose $z \in \sigma(A,a)$. Then $\gamma_n(A,a,z) = 0$ and $(z-a)^m$ is not invertible in A. Thus, either $p_k(z-a)^m$ is not invertible in p_kAp_k for some $k \in I$, or, $p_j(\lambda-a)^m$ is invertible in p_jAp_j with inverses $p_jc_jp_j \forall\, j \in I$, but $\sup_{j\in I}\|p_jc_jp_j\| = \infty$, using (7). In either case, we have $\inf_{j\in I} \gamma_n(p_jAp_j, p_ja, z) = 0$.

3. Observe that

$$\begin{aligned}
&\lambda \in \Lambda_{n,\epsilon}(A,a)\\
&\iff \gamma_n(A,a,\lambda) \le \epsilon\\
&\iff \inf_{j\in I} \gamma_n(p_jAp_j, p_ja, z) \le \epsilon\\
&\iff \exists\, k \in I \text{ such that } \gamma_n(p_kAp_k, p_ka, z) \le \epsilon \text{ or, } \inf_{j\in I} \gamma_n(p_jAp_j, p_ja, z) = \epsilon\\
&\iff \lambda \in \Lambda_{n,\epsilon}(p_kAp_k, p_ka) \text{ or, } \inf_{j\in I} \gamma_n(p_jAp_j, p_ja, z) = \epsilon.
\end{aligned}$$

□

Remark 4 Note that if I is a finite set, then the second set on the right hand side of (8) will be empty and (8) will coincide with the results given in Lemma 1(2).

Now we provide an example where the conditions in the hypotheses of the Theorem 4 are satisfied and consequently the results of Theorem 4 hold.

Example 1 Let $(X_j, \|\cdot\|_j)$ be an at most family of Banach spaces. For $1 \le r < \infty$, let

$$X^r := \{x = \{x_j\} : x_j \in X_j \text{ with } \sum_{j\in I} \|x_j\|_j^r < \infty\}, \text{ and}$$

$$X^\infty := \{x = \{x_j\} : x_j \in X_j \text{ with } \sup_{j\in I}\|x_j\|_j < \infty\}.$$

The co-ordinatewise linear operations make X^r and X^∞ vector spaces. If $x = \{x_j\}$, then define the norms by

$$\|x\|_r = \left(\sum_{j\in I} \|x_j\|_j^r\right)^{\frac{1}{r}}, \ 1 \le r < \infty, \ \text{ and } \|x\|_\infty = \sup_{j\in I}\|x_j\|_j.$$

Then for $1 \le r \le \infty$, $(X^r, \|\cdot\|_r)$ becomes a Banach space, known as the direct sum of Banach spaces (see [5]). Let $1 \le r < \infty$. For each $j \in I$, define $P_j : X^r \to X^r$ by

$$P_j(x_1, x_2, \ldots, \boxed{x_j}, x_{j+1}, \ldots) = (0, \ldots, \boxed{x_j}, 0, \ldots)$$

Then $P_j^2 = P_j$ and $\|P_j\| = 1$ for all $j \in I$ and for $x = \{x_j\} \in X^r$,

$$\|P_j x\|_r = \|x_j\|_j, \text{ and hence } \|x\|_r^r = \left(\sum_{j\in I} \|P_j(x)\|_r^r\right).$$

In this case we have $x = \sum_{j\in I} P_j(x)\ \forall x \in X^r$. Let $T \in B(X^r)$ with $TP_j = P_jT$ $\forall j \in I$. Suppose $\alpha = \sup_{j\in I}\|P_jT\|$. Clearly, $\alpha \le \|T\|$. Further, for $x \in X^r$,

$$\begin{aligned}
\|Tx\|_r^r &= \sum_{j\in I} \|P_jT(x)\|_r^r \\
&= \sum_{j\in I} \|TP_jP_j(x)\|_r^r \\
&\le \sum_{j\in I} \big(\|TP_j\|\|P_j(x)\|_r\big) \\
&\le \alpha^r \sum_{j\in I} (\|P_j(x)\|_r^r) \\
&= \alpha^r \|x\|_r^r.
\end{aligned}$$

Thus $\|Tx\|_r \le \alpha\|x\|_r\ \forall x \in X^r$. Hence $\|T\| = \alpha = \sup_{j\in I}\|P_jT\|$ and consequently (6) is satisfied.

Note that $P_jB(X^r)P_j = B(P_j(X^r))$. Now let $T_j = P_jT\ |_{P_j(X^r)}\ \forall j \in I$. Then $T_j \in B(P_j(X^r))\ \forall j \in I$. Suppose each T_j is invertible with inverse S_j and $\sup_{j\in I}\|S_j\| = \beta$ (say) $< \infty$. Then we show that T is invertible in $B(X^r)$ and $\|T^{-1}\| = \beta$.

Now, for any $y \in X^r$, we observe that

$$\sum_{j\in I}\|S_jP_j(y)\|_j^r \le \beta \sum_{j\in I} \|P_j(y)\|_j^r = \beta\|y\|_r^r < \infty.$$

Thus we can define the operator S, which is the direct sum of the operators S_j, i.e., $S(y) = \sum\limits_{j\in I} S_j P_j(y)\ \forall y \in X^r$. The sum is understood pointwise in the sense of strong convergence. Note that $\|S(y)\|_r^r \le \beta \|y\|_r^r$ and hence $S \in B(X^r)$.

We show that $\|S\| = \beta$. To see this, let $\epsilon > 0$. There exists $k \in I$ and $x_0 \in P_k(X^r)$ with $\|x_0\| = 1$ such that $\beta - \epsilon < \|S_k(x_0)\|_k$. Now, since $x_0 = p_k(x_0)$, we get

$$\|S(x_0)\|_r^r = \sum_{j\in I} \|S_j P_j(x_0)\|_j^r = \|S_k(x_0)\|_k^r > (\beta - \epsilon)^r.$$

This holds for every $\epsilon > 0$. Thus it follows that $\|S\| = \beta$.

Now Range$(S_j) = P_j(X^r)$ and $P_j P_k = 0\ \forall k \ne j$. Hence $P_k S = S_k P_k\ \forall\, k$. Further

$$\begin{aligned} ST(x) &= \sum_{j\in I} S_j P_j(Tx) = \sum_{j\in I} S_j P_j T P_j(x) = \sum_{j\in I} S_j T_j P_j(x) \\ &= \sum_{j\in I} P_j(x) = x\ \forall\, x \in X^r. \end{aligned}$$

Again, for $x \in X^r$, we have

$$\begin{aligned} TS(x) = \sum_{j\in I} T S_j P_j(x) &= \sum_{j\in I} T P_j S_j P_j(x) \\ &= \sum_{j\in I} T_j S_j P_j(x) \\ &= \sum_{j\in I} P_j(x) \\ &= x. \end{aligned}$$

Consequently, T is invertible, $T^{-1} = S$ and $\|T^{-1}\| = \|S\| = \sup\limits_{j\in I} \|S_j\|$. Note that T can be expressed as a direct sum of the operators T_j, $j \in I$. Now the hypotheses of Theorem 4 are satisfied. Hence

$$\sigma(B(X^r), T) = \bigcup_{j\in I} \sigma(B(P_j(X^r)), T_j) \cup \{\lambda \in \mathbb{C} : \sup_{j\in I} \|P_j K_j P_j\| = \infty\},$$

where $P_j K_j P_j$ is the inverse of $P_j(\lambda - T)\ |_{P_j(X^r)}$ for each $j \in I$, and

$$\begin{aligned} &\Lambda_{n,\epsilon}(B(X^r), T) \\ &\quad = \bigcup_{j\in I} \Lambda_{n,\epsilon}(B(P_j(X^r), T_j) \cup \left\{ \lambda \in \mathbb{C} : \inf_{j\in I} \gamma_n(B(P_j(X^r)), T_j, \lambda) = \epsilon \right\} \end{aligned}$$

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