Separation Dimension and Sparsity

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Abstract

The *separation dimension* $\pi(G)$ of a hypergraph G is the smallest natural number k for which the vertices of G can be embedded in $\mathbb{R}^k$ so that any pair of disjoint edges in G can be separated by a hyperplane normal to one of the axes. Equivalently, it is the cardinality of a smallest family $\mathcal{F}$ of total orders of $V(G)$, such that for any two disjoint edges of G, there exists at least one total order in $\mathcal{F}$ in which all the vertices in one edge precede those in the other.

Separation dimension is a monotone parameter; adding more edges cannot reduce the separation dimension of a hypergraph. In this article we discuss the influence of separation dimension and edge-density of a graph on one another. On one hand, we show that the maximum separation dimension of a k-degenerate graph on n vertices is $O(k \lg \lg n)$ and that there exists a family of 2-degenerate graphs with separation dimension $\Omega(\lg \lg n)$. On the other hand, we show that graphs with bounded separation dimension cannot be very dense. Quantitatively, we prove that n-vertex graphs with separation dimension s have at most $3(4 \lg n)^{s-2} \cdot n$ edges. We do not believe that this bound is optimal and give a question and a remark on the optimal bound.

Keywords: separation dimension, edge density, degeneracy.

1 Introduction

Let $\sigma : U \to [n]$ be a permutation of elements of an n-set U and let $\prec_\sigma$ denote the associated total order. For two disjoint subsets A, B of U, we say $A \prec_\sigma B$ when every element of A precedes every element of B in σ, i.e., $\sigma(a) < \sigma(b),\ \forall (a,b) \in A \times B$. We say that σ *separates* A and B if either $A \prec_\sigma B$ or $B \prec_\sigma A$. For two subsets A, B of U, we say $A \preceq_\sigma B$ when $A \setminus B \prec_\sigma A \cap B \prec_\sigma B \setminus A$.

Definition 1.1. A family $\mathcal{F}$ of permutations of $V(H)$ is *pairwise suitable* for a hypergraph H if, for every two disjoint edges $e, f \in E(H)$, there exists a permutation $\sigma \in \mathcal{F}$ which separates e and f. The cardinality of a smallest family of permutations that is pairwise suitable for H is the *separation dimension* of H and is denoted by $\pi(H)$.

A family $\mathcal{F} = \{\sigma_1, \ldots, \sigma_k\}$ of permutations of a set V can be seen as an embedding of V into $\mathbb{R}^k$ with the i-th coordinate of $v \in V$ being $\sigma_i(v)$. Similarly, given any embedding of V in $\mathbb{R}^k$, we can construct k permutations by projecting the points onto each of the k axes and then reading them along each axis, breaking any ties arbitrarily. From this, it is easy to see that $\pi(H)$ is the smallest natural number k so that the vertices of H can be embedded into $\mathbb{R}^k$ such that any two disjoint edges of H can be separated by a hyperplane normal to one of the axes. This motivates us to call such an embedding a *separating embedding* of H and $\pi(H)$ the *separation dimension* of H.

The notion of separation dimension was introduced by the authors in [2][1] and further studied in [1, 3]. Apart from its naturalness, a major motivation to study this notion of separation is its interesting connection with a certain well studied geometric representation of graphs. The *boxicity* of a graph G is the minimum natural number k for which G can be represented as an intersection graph of axis-parallel boxes in $\mathbb{R}^k$. The separation dimension of a hypergraph H is equal to the boxicity of the intersection graph of the edge set of H, i.e., the line graph of H [3].

1.1 Related notions

Families of permutations which satisfy some type of "separation" properties have been long studied in combinatorics. One of the early examples is the work of Ben Dushnik in 1950 where he introduced the notion of *k-suitability* [6]. A family $\mathcal{F}$ of permutations of $[n]$ is *k-suitable* if, for every $(k-1)$-set $A \subseteq [n]$ and for every $b \in [n] \setminus A$, there exists a $\sigma \in \mathcal{F}$ such that $A \prec_\sigma \{b\}$. Let $N(n,k)$ denote the cardinality of a smallest family of permutations that is k-suitable for $[n]$. In 1972, Spencer [15] proved that $\lg\lg n \leq N(n,3) \leq N(n,k) \leq k2^k \lg\lg n$. Fishburn and Trotter, in 1992, defined the *dimension* of a hypergraph on the vertex set $[n]$ to be the minimum size of a family $\mathcal{F}$ of permutations of $[n]$ such that every edge of the hypergraph is an intersection of *initial segments* of $\mathcal{F}$ [8]. It is easy to see that an edge e is an intersection of initial segments of $\mathcal{F}$ if and only if for every $v \in [n] \setminus e$, there exists a permutation $\sigma \in \mathcal{F}$ such that $e \prec_\sigma \{v\}$. It is interesting to note that, according to this definition, $N(n,k)$ is the dimension of the complete $(k-1)$-uniform hypergraph on n vertices. The quantity $N(n,3)$, which is thus the dimension of the n-vertex complete graph, and which is also the (poset) dimension of the inclusion poset of 1 and 2 sized subsets of $[n]$ is a parameter that appears in many combinatorial problems, including the one we study here. Very tight estimates which can determine the exact value of $N(n,3)$ for almost all n were given by Hoşten and Morris in 1999 by finding a nice equivalence of this problem to a variant of the Dedekind problem [12].

Füredi, in 1996, studied the notion of 3-*mixing* family of permutations [9]. A family $\mathcal{F}$ of permutations of $[n]$ is called 3-mixing if for every 3-set $\{a,b,c\} \subseteq [n]$ and a designated element a in that set, one of the permutations in $\mathcal{F}$ places the element a between b and c. It is clear that a is between b and c in a permutation σ if and only if $\{a,b\} \preceq_\sigma \{a,c\}$ or $\{a,c\} \preceq_\sigma \{a,b\}$. Such families of permutations with small sizes have found applications in showing upper bounds for many combinatorial parameters including poset dimension [13], product dimension [10] and boxicity [5].

[1] Some of our initial results on this topic including some of the results reported here are available as a preprint in arXiv [4].

1.2 Our results

The separation dimension of K_n, the complete graph on n vertices is $\Theta(\lg n)$ [3]. It is easy to see that separation dimension is a monotone property, i.e., $\pi(G') \leq \pi(G)$ if G' is a subgraph of G. So it is interesting to check whether the separation dimension of sparse graph families can be much lower than $\lg n$. Here, by a *sparse family of graphs*, we mean a family of graphs in which the number of edges is linear in the number of vertices.

Sparsity of a graph family, as we consider it here, is equivalent to the restriction that the graphs in the family have bounded average degree. It is easy to see that globally sparse graphs with a small dense subgraph can have large separation dimension. For example if we consider an n vertex graph which is a disjoint union of a complete graph on $\lfloor\sqrt{n}\rfloor$ vertices and remaining isolated vertices, it has at most $n/2$ edges, but has a separation dimension in $\Omega(\lg n)$ due to the clique. Hence sparsity is needed across all subgraphs in order to hope for a better upper bound on separation dimension. One common way of ensuring local sparsity of a graph family is to demand that the degeneracy of the graphs in the family be bounded.

Definition 1.2 (Degeneracy)**.** For a non-negative integer k, a graph G is *k-degenerate* if the vertices of G can be linearly ordered in such a way that every vertex is succeeded by at most k of its neighbours. The least number k such that G is k-degenerate is called the *degeneracy* of G and any such enumeration is referred to as a *degeneracy order* of $V(G)$.

For example, trees and forests are 1-degenerate and planar graphs are 5-degenerate. Series-parallel and outerplanar graphs are 2-degenerate. Graphs of treewidth t are t-degenerate. It is easy to verify that if the maximum average degree over all subgraphs of a graph G is d, then G is $\lfloor d\rfloor$-degenerate. A $\lfloor d\rfloor$-degeneracy order of G can be obtained by recursively picking out a minimum degree vertex from G. It is also easy to see that any subgraph of a k-degenerate graph has average degree at most $2k$.

In this paper we establish the following upper bound on the separation dimension of k-degenerate graphs and thereby give an affirmative answer to our question under a restricted but necessary condition of sparsity.

Theorem 1.3. *For a k-degenerate graph G on n vertices,*

$$\pi(G) \leq 2k\left(N(n,3)+3\right),$$

where $N(n,3)$ is the minimum cardinality of a family of permutations that is 3-suitable for $[n]$.

Remark. By the bound $N(n,3) \leq \lg\lg n + \frac{1}{2}\lg\lg\lg n + \frac{1}{2}\lg\pi + 1 + o(1)$ obtainable from the equivalence established by Hoşten and Morris [12], it follows that

$$\pi(G) \leq k\left(2\lg\lg n + \lg\lg\lg n + \lg\pi + 8 + o(1)\right).$$

We prove Theorem 1.3 by decomposing G into $2k$ star forests and using 3-suitable permutations of the stars in every forest and the leaves in every such star simultaneously. The proof is given in Section 2.2. We show that the $\lg\lg n$ factor in Theorem 1.3 cannot be improved in general by estimating the separation dimension of a fully subdivided clique.

Definition 1.4 (Fully subdivided graphs)**.** A graph G' is called a *subdivision* of a graph G if G' is obtained from G by replacing a subset of edges of G with independent paths between their ends such that none of these new paths has an inner vertex on another path or in G. A subdivision of G where every edge of G is replaced by a k-length path is denoted as $G^{1/k}$. The graph $G^{1/2}$ is called *fully subdivided G*.

Notice that $G^{1/2}$ is the comparability graph of the incidence poset of G. It is easy to see that $G^{1/2}$ is a 2-degenerate graph for any graph G. A 2-degeneracy order can be obtained by picking out all the vertices introduced by the subdivision first.

Theorem 1.5. *Let $K_n^{1/2}$ denote the graph obtained by fully subdividing K_n. Then, $\frac{1}{2}f(n) \leq \pi(K_n^{1/2}) \leq g(n)$, where both $f(n)$ and $g(n)$ are $\lg\lg(n-1) + \left(\frac{1}{2} + o(1)\right)\lg\lg\lg(n-1)$.*

We establish the lower bound by first using the Erdős-Szekeres Theorem to extract a large enough set of vertices of the underlying K_n that are ordered essentially the same by every permutation in the selected family and then showing that separating the edges incident on those vertices can be modelled as a problem of finding a realiser for a canonical open interval order (cf. Definition 2.6 in Section 2.4) of the same size. The details are given in Section 2.4. The upper bound follows from the next result.

Theorem 1.6. *For a graph G with chromatic number $\chi(G)$,*

$$\pi(G^{1/2}) \leq \lg\lg(\chi(G)-1) + \left(\frac{1}{2} + o(1)\right)\lg\lg\lg(\chi(G)-1).$$

We establish this by associating with every graph G an interval order whose dimension (cf. Definition 2.4 in Section 2.3) is at least $\pi(G^{1/2})$ and whose height is less than the chromatic number of G. The result then follows from an estimate on the dimension of interval orders due to Füredi, Hajnal, Rödl and Trotter [11]. The details are given in Section 2.3. It follows from the previous result on $\pi(K_n^{1/2})$ that the above upper bound is at most a 2-factor off.

Even though the study of $K_n^{1/2}$ reveals that local sparsity alone cannot ensure boundedness of separation dimension, many sparse graph families with further structure indeed have a bounded separation dimension. A useful result in this direction is that $\pi(G) \in O(\chi_a(G))$, where $\chi_a(G)$ denotes the acyclic chromatic number of the graph G [3]. Hence any graph family with bounded acyclic chromatic number will have bounded separation dimension too[2]. This includes many sparse families of graphs like bounded degree graphs, bounded treewidth graphs, graphs with bounded Euler genus, graphs with no $K_{\eta+1}$ minor and so on. It follows from results in the literature on the acyclic chromatic number that the separation dimension of a graph G is in $O(d^{4/3})$, $O(t)$, $O(g^{4/7})$ and $O(\eta^2 \lg \eta)$, where d is the maximum degree, t is the treewidth, g is the Euler genus and η is the Hadwiger number (the size of the largest complete minor) of G. Some of these families were studied separately in an attempt to improve the above bounds and make them as tight as possible. For example, trees and outerplanar graphs have maximum separation dimension 2; series-parallel and planar graphs have maximum separation dimension 3 [3]; and the maximum separation dimension of degree d-bounded graphs is at most $2^{9\lg^\star d}d$ and at least $\lceil d/2 \rceil$ [1].

Interestingly and a bit frustratingly, most of these strong sparsity conditions discussed above (maximum degree, treewidth, Euler genus, or Hadwiger number) are not necessary for a graph family to have bounded separation dimension. Hence all these results beg for an investigation into the extremal and structural properties of the family of graphs with bounded separation dimension - like how dense can the family be, is its acyclic chromatic number bounded and so on. We do not even know whether this family has bounded chromatic number.

It follows easily from the characterisation of graphs with separation dimension at most 1 in [3] that they are 2-degenerate. It is not difficult to see that if a graph G has separation dimension at most 2, then it is planar. Indeed, the natural embedding of G into $\mathbb{R}^2$ determined by the family of two permutations which is pairwise-suitable for G has the property that two

[2]Notice that bipartite graphs, $K_n^{1/2}$ as we see here for instance, can have unbounded separation dimension and hence separation dimension cannot be bounded above by a function of chromatic number alone.

disjoint edges do not cross. We can modify this representation, for instance, using a set of rationally independent real numbers as the ordinates, to ensure that two edges of G which share a common vertex intersect only at their common end-point. This gives a planar drawing of G. Thus graphs with separation dimension at most 2 are planar and hence 5-degenerate. This prompts us to ask the following question.

Open problem 1.7. *Does there exist a function $f : \mathbb{N} \to \mathbb{R}$ such that every graph with separation dimension s is $f(s)$-degenerate?*

We do not know the answer to the above problem yet. We can show that graphs with bounded separation dimension cannot be very dense. In Section 2.5, we prove

Theorem 1.8. *Every n-vertex graph with separation dimension s has at most $3(4 \lg n)^{s-2} n$ edges.*

In Section 2.5, we also prove

Theorem 1.9. *For the d-dimensional hypercube Q_d,*

$$\pi(K_d^{1/2}) \leq \pi(Q_d) \leq N(d, 3).$$

Since the degeneracy of Q_d is d and $N(d, 3)$ is $\lg \lg d + (1/2 + o(1)) \lg \lg \lg d$, we can see that if the answer to Open problem 1.7 if affirmative, then f is at least doubly exponential.

2 Proofs

2.1 Notational note

All graphs considered in this article are finite, simple and undirected. The vertex set and edge set of a graph G are denoted respectively by $V(G)$ and $E(G)$. For any finite positive integer n, we shall use $[n]$ to denote the set $\{1, \ldots, n\}$. We use the one-line notation for permutations, i.e., a permutation $\sigma : [n] \to [n]$ is denoted by $(\sigma(1), \ldots, \sigma(n))$. The logarithm of any positive real number x to the base 2 is denoted by $\lg(x)$, while $\lg^\star(x)$ denotes the iterated logarithm of x to the base 2, i.e. the number of times the logarithm function (to the base 2) should be applied so that the result is less than or equal to 1.

2.2 Upper bound: k-degenerate graphs

For any non-negative integer n, a *star* S_n is a rooted tree on $n + 1$ nodes with one root and n leaves connected to the root. A *star forest* is a disjoint union of stars.

Definition 2.1. The *arboricity* of a graph G, denoted by $\mathcal{A}(G)$, is the minimum number of spanning forests whose union covers all the edges of G. The *star arboricity* of a graph G, denoted by $\mathcal{S}(G)$, is the minimum number of spanning star forests whose union covers all the edges of G.

Clearly, $\mathcal{S}(G) \geq \mathcal{A}(G)$ from definition. Furthermore, since any tree can be covered by two star forests, $\mathcal{S}(G) \leq 2\mathcal{A}(G)$.

For the sake of completeness, we give a proof for the following already-known lemma on star arboricity of k-degenerate graphs (Definition 1.2).

Lemma 2.2. *For a k-degenerate graph G, $\mathcal{S}(G) \leq 2k$.*

Proof. By following the degeneracy order, the edges of G can be oriented acyclically such that each vertex has an out-degree at most k. Now the edges of G can be partitioned into k spanning forests by choosing a different forest for each outgoing edge from a vertex. Thus, $\mathcal{A}(G) \leq k$ and $\mathcal{S}(G) \leq 2k$. □

With this we now give a proof of our first result.

Proof of Theorem 1.3.

Statement. For a k-degenerate graph G on n vertices,

$$\pi(G) \leq 2k\left(N(n,3)+3\right),$$

where $N(n,3)$ is the minimum cardinality of a family of permutations that is 3-suitable for $[n]$.

Proof. Let $r = N(n,3)$ and let $\mathcal{T} = \{\tau^1, \ldots, \tau^r\}$ be a family of permutations that is 3-suitable for $V(G)$. Recall that a family $\mathcal{T}$ of permutations of V is called 3-suitable if for every three distinct vertices $a, b, c \in V(G)$ there exists a permutation $\tau \in \mathcal{T}$ such that $\{a,b\} \prec_\tau \{c\}$.

By Lemma 2.2, we can partition the edges of G into a collection of $2k$ spanning star forests $\{C_1, \ldots, C_{2k}\}$. For each star forest C_i, we construct a family $\mathcal{F}_i = \{\sigma_i^1, \ldots, \sigma_i^r, \alpha_i, \beta_i, \gamma_i\}$ of permutations of $V(G)$ as follows. In σ_i^j, consider each star in C_i as a block and arrange the stars according to the order of their roots in τ^j and then within each star the leaves are arranged according to their order in τ^j followed by the root. The permutation α_i is obtained from σ_i^1 by reversing the order of leaves within each star without changing the order among the stars. The permutation β_i is obtained from σ_i^1 by reversing the order among the stars without changing the order of vertices within a star. Finally, the permutation γ_i is obtained from σ_i^1 by reversing both the order of leaves within a star and the order among the stars. Notice that in all the permutations in $\mathcal{F}_i$, the vertices of each star in C_i appear consecutively with the root being the rightmost.

Claim 1. *$\mathcal{F} = \bigcup_{i=1}^{2k} \mathcal{F}_i$ is a pairwise-suitable family of permutations for G.*

Let $\{a,b\}, \{c,d\}$ be two disjoint edges in G. Let C_i be the star forest which contains the edge $\{a,b\}$. We will show that at least one of the permutations in $\mathcal{F}_i$ will separate these two edges. Since the edge $\{a,b\}$ is present in C_i, the vertices a and b belong to the same star, say S_a, of C_i. Without loss of generality, we can assume that a is the root of S_a. Let x and y be the roots of the stars S_x and S_y in C_i which contain the vertices c and d, respectively. It is not necessary that the vertices a, x and y are distinct.

If neither c nor d is in S_a then $\{a,b\}$ and $\{c,d\}$ are separated in σ_i^j where τ^j is a permutation in which a succeeds both x and y. If both c and d are in S_a, that is, if $x = y = a$, then the two edges are separated in σ_i^j where τ^j is a permutation in which b succeeds c and d. If exactly one of c or d, say c, is in S_a, that is, $y \neq x = a$, then $\{a,b\}$ is separated from $\{c,d\}$ in one of σ_i^1, α_i, β_i or γ_i.

This proves Claim 1 and, as $|\mathcal{F}| \leq 2k(r+3)$, the theorem. □

2.3 Upper bound: Fully subdivided graphs

In this section we establish an upper bound for $\pi(G^{1/2})$ in terms of $\chi(G)$, the chromatic number of G. With the definitions recalled below, we do this by constructing an interval order based on G of height $\chi(G)-1$ and then showing that its poset dimension is an upper bound on $\pi(G^{1/2})$. We need some more definitions and notation before proceeding.

Definition 2.3 (Poset dimension)**.** Let $(\mathcal{P}, \lhd)$ be a poset (partially ordered set). A *linear extension* L of $\mathcal{P}$ is a total order which satisfies $(x \lhd y \in \mathcal{P}) \implies (x \lhd y \in L)$. A *realiser* of $\mathcal{P}$ is a set of linear extensions of $\mathcal{P}$, say $\mathcal{R}$, which satisfy the following condition: for any two distinct elements x and y, $x \lhd y \in \mathcal{P}$ if and only if $x \lhd y \in L, \forall L \in \mathcal{R}$. The *poset dimension* of $\mathcal{P}$, denoted by $\dim(\mathcal{P})$, is the minimum integer k such that there exists a realiser of $\mathcal{P}$ of cardinality k.

Definition 2.4 (Interval dimension)**.** An *open interval* on the real line, denoted as (a, b), where $a, b \in \mathbb{R}$ and $a < b$, is the set $\{x \in \mathbb{R} : a < x < b\}$. For a collection C of open intervals on the real line the partial order $(C, \lhd)$ defined by the relation $(a, b) \lhd (c, d)$ if $b \leq c$ in $\mathbb{R}$ is called the *interval order* corresponding to C. The poset dimension of this interval order $(C, \lhd)$ is called the *interval dimension* of C and is denoted by $\dim(C)$.

The major part of our proof of Theorem 1.6 is the following lemma.

Lemma 2.5. *For any graph G and a permutation σ of $V(G)$, let $C_{G,\sigma}$ denote the collection of open intervals $(\sigma(u), \sigma(v)), \{u, v\} \in E(G), u \prec_\sigma v$. Then,*

$$\pi(G^{1/2}) \leq \min_{\sigma} \dim(C_{G,\sigma}) + 2,$$

where the minimisation is done over all possible permutations σ of $V(G)$.

Proof. Let σ be any permutation of $V(G)$. We relabel the vertices of G so that $v_1 \prec_\sigma \cdots \prec_\sigma v_n$, where $n = |V(G)|$. For every edge $e = \{v_i, v_j\} \in E(G), i < j$, the new vertex in $G^{1/2}$ introduced by subdividing e is denoted as u_{ij}. For a new vertex u_{ij}, its two neighbours, v_i and v_j will be respectively called the *left neighbour* and *right neighbour* of u_{ij}. We call an edge of the form $\{v_i, u_{ij}\}$ as a *left edge* and one of the form $\{u_{ij}, v_j\}$ as a *right edge*.

Let $\mathcal{R} = \{L_1, \ldots, L_d\}$ be a realiser for $(C_{G,\sigma}, \lhd)$ such that $d = \dim(C_{G,\sigma})$. For each total order $L_p, p \in [d]$, we construct a permutation σ_p of $V(G^{1/2})$ as follows. First, the subdivided vertices are ordered from left to right as the corresponding intervals are ordered in L_p, i.e., $u_{ij} \prec_{\sigma_p} u_{kl} \iff (i, j) \prec_{L_p} (k, l)$. Next, the original vertices are introduced into the order one by one as follows. The vertex v_1 is placed as the leftmost vertex. Once all the vertices v_i, $i < j$, are placed, we place v_j at the leftmost possible position so that $v_{j-1} \prec_{\sigma_p} v_j$ and $u_{ij} \prec_{\sigma_p} v_j, \forall i < j$. This ensures that $v_j \prec_{\sigma_p} u_{jk}, \forall k > j$ because $u_{ij'} \prec_{\sigma_p} u_{jk}, \forall j' \leq j$ (Since $(i, j') \lhd (j, k)$). Now we construct two more permutations σ_{d+1} and σ_{d+2} as follows. In both of them, first the original vertices are arranged in the order $v_1, \ldots, v_n$. In σ_{d+1}, the subdivided vertices are placed (in any order) immediately after its left neighbour, i.e., $v_i \prec_{\sigma_{d+1}} u_{ij} \prec_{\sigma_{d+1}} v_{i+1}$ for all $\{i, j\} \in E(G)$. In σ_{d+2}, the subdivided vertices are placed (in any order) immediately before its right neighbour, i.e., $v_{j-1} \prec_{\sigma_{d+2}} u_{ij} \prec_{\sigma_{d+2}} v_j$ for all $\{i, j\} \in E(G)$. Notice that in all the permutations so far constructed, the left (right) neighbour of every subdivided vertex is placed to its left (right).

We complete the proof by showing that $\mathcal{F} = \{\sigma_1, \ldots, \sigma_{d+2}\}$ is pairwise suitable for $G^{1/2}$ by analysing the following cases. Any two disjoint left edges are separated in σ_{d+1} and any two disjoint right edges are separated in σ_{d+2}. If $(i, j) \lhd (k, l)$, then every pair of disjoint edges among those incident on u_{ij} or u_{kl} are separated in every permutation in $\mathcal{F}$. Hence the only non-trivial case is when we have a left edge $\{v_i, u_{ij}\}$ and a right edge $\{u_{kl}, v_l\}$ such that $(i, j) \cap (k, l) \neq \emptyset$. Since (i, j) and (k, l) are incomparable in $(C_{G,\sigma}, \lhd)$, there exists one permutation $\sigma_p, p \in [d]$ such that $u_{ij} \prec_{\sigma_p} u_{kl}$ (and another permutation $\sigma_q, q \in [d]$ such that $u_{kl} \prec_{\sigma_q} u_{ij}$). Since v_i is before u_{ij} and v_l is after u_{kl} in every permutation, σ_p separates $\{v_i, u_{ij}\}$ from $\{u_{kl}, v_l\}$. □

Proof of Theorem 1.6

The *height* of a partial order is the size of a largest chain in it. It was shown by Füredi, Hajnal, Rödl and Trotter [11] that the maximum dimension of an interval order of height h

is $\lg\lg h + (\frac{1}{2} + o(1))\lg\lg\lg h$ (see also Theorem 9.6 in [16]). A proof of Theorem 1.6 is now immediate.

Statement. For a graph G with chromatic number $\chi(G)$,

$$\pi(G^{1/2}) \leq \lg\lg(\chi(G) - 1) + \left(\frac{1}{2} + o(1)\right)\lg\lg\lg(\chi(G) - 1).$$

Proof. Let $V_1, \ldots, V_{\chi(G)}$ be the colour classes of an optimal proper colouring of G. Let σ be a permutation of $V(G)$ such that $V_1 \prec_\sigma \cdots \prec_\sigma V_{\chi(G)}$. Now it is easy to see that the longest chain in $(C_{G,\sigma}, \lhd)$ is of length at most $\chi(G) - 1$. Hence the statement follows from the result of Füredi et al. [11] and Lemma 2.5 above. □

2.4 Lower bound: Fully subdivided clique

It easily follows from Theorem 1.6 that $\pi(K_n^{1/2}) \in O(\lg\lg n)$. In this section we prove that $\pi(K_n^{1/2}) \in \Omega(\lg\lg n)$. We give a brief outline of the proof below. (Definitions of the new terms are given before the formal proof.)

First, we use the Erdős-Szekeres Theorem [7] to argue that for any family $\mathcal{F}$ of permutations of $V(K_n^{1/2})$, with $|\mathcal{F}| < \frac{1}{2}\lg\lg n$, a subset V' of original vertices of $K_n^{1/2}$, with cardinality $n' = |V'| \approx 2^{\sqrt{\lg n}}$, is ordered essentially in the same way by every permutation in $\mathcal{F}$. Since the ordering of the vertices in V' are fixed, the only way for $\mathcal{F}$ to realise pairwise suitability among the edges in the subdivided paths between vertices in V' is to find suitable positions for the new vertices (those introduced by subdivisions) inside the fixed order of V'. We then show that this amounts to constructing a realiser for the canonical open interval order $(C_{n'}, \lhd)$ and hence $|\mathcal{F}|$, in this case, is bounded below by the poset dimension of $(C_{n'}, \lhd)$. It follows quite easily from existing literature that the poset dimension of this canonical open interval order is at least $\frac{1}{2}\lg\lg n$ for large n.

Definition 2.6 (Canonical open interval order)**.** For an integer $n \geq 3$, let $C_n = \{(a,b) : a, b \in [n], a < b\}$ be the collection of all the $\binom{n}{2}$ open intervals which have their endpoints in $[n]$. Then $(C_n, \lhd)$, the interval order corresponding to the collection C_n, is called the *canonical open interval order.*

Usually, in literature, the canonical interval order is defined over non-degenerate closed intervals. For a positive integer n, let $I_n = \{[a,b] : a, b \in [n], a < b\}$ be the collection of all the $\binom{n}{2}$ non-degenerate closed intervals which have their endpoints in $[n]$. The poset $(I_n, \lhd')$, where $[i,j] \lhd' [k,l] \iff j < k$ is called the *canonical (closed) interval order*. It is easy to see, by examining the map $[i,j] \mapsto (i, j+1)$, that $(I_{n-1}, \lhd')$ is isomorphic to the poset obtained from $(C_n, \lhd)$ by removing all the unit-length intervals. Therefore $\dim(C_n) \geq \dim(I_{n-1})$. It was established by Füredi, Hajnal, Rödl and Trotter [11] that the dimension of $(I_n, \lhd')$ is bounded below by the chromatic number of the double shift graph $G(n,3)$ which in turn is equal to the smallest number t for which there are at least n anti-chains in the lattice of subsets of $[t]$ - the inverse of the classic Dedekind problem. Due to the work of Kleitman and Markovsky on the Dedekind problem [14], we know that the chromatic number of the double shift graph $G(n,3)$ is at least $\lg\lg n + \frac{1}{2}\lg\lg\lg n + \frac{1}{2}\lg\frac{\pi}{2} + o(1)$. Hence we can conclude that $\dim(C_n) \geq \lg\lg(n-1) + \frac{1}{2}\lg\lg\lg(n-1) + \frac{1}{2}\lg\frac{\pi}{2} + o(1)$.

Proof of Theorem 1.5

Statement. Let $K_n^{1/2}$ denote the graph obtained by fully subdividing K_n. Then, $\frac{1}{2}f(n) \leq \pi(K_n^{1/2}) \leq g(n)$, where both $f(n)$ and $g(n)$ are $\lg\lg(n-1) + \left(\frac{1}{2} + o(1)\right)\lg\lg\lg(n-1)$.

Proof. The upper bound follows from Theorem 1.6. So it suffices to show the lower bound.

Let $m = \lfloor \lg \lg(n-1) \rfloor$ and hence $2^{2^m} \le n-1 < 2^{2^{m+1}}$. Let $v_1, \ldots, v_n$ denote the *original vertices* (the vertices of degree $n-1$) in $K_n^{1/2}$ and let u_{ij}, $i, j \in [n], i < j$, denote the new vertex of degree 2 introduced when the edge $\{i, j\}$ of K_n was subdivided. Let $\mathcal{F}$ be a family of permutations that is pairwise suitable for $K_n^{1/2}$ such that $|\mathcal{F}| = r = \pi(K_n^{1/2})$. By the Erdős-Szekeres Theorem [7], we know that if τ and τ' are two permutations of $[n^2+1]$, then there exists some $X \subseteq [n^2+1]$ with $|X| = n+1$ such that the permutations τ and τ' when restricted to X are the same or reverse of each other. Then by repeated application of the Erdős-Szekeres Theorem, we can see that there exists a set X of $p = 2^{2^{m-r+1}} + 1$ original vertices of $K_n^{1/2}$ such that, for each $\sigma, \sigma' \in \mathcal{F}$, the permutation of X obtained by restricting σ to X is the same or reverse of the permutation obtained by restricting σ' to X. Without loss of generality, let $X = \{v_1, \ldots, v_p\}$ such that, for each $\sigma \in \mathcal{F}$, either $v_1 \prec_\sigma \cdots \prec_\sigma v_p$ or $v_p \prec_\sigma \cdots \prec_\sigma v_1$.

Next we "massage" $\mathcal{F}$ to give it two nice properties without changing its cardinality or sacrificing its pairwise suitability for $K_n^{1/2}$. Note that if a family of permutations is pairwise suitable for a graph then the family retains this property even if any of the permutations in the family is reversed. Hence we can assume the following property without loss of generality.

Property 1. $v_1 \prec_\sigma \cdots \prec_\sigma v_p, \forall \sigma \in \mathcal{F}$.

Consider any $i, j \in [p], i < j$. For each $\sigma \in \mathcal{F}$, it is safe to assume that $v_i \prec_\sigma u_{ij} \prec_\sigma v_j$. Otherwise, we can modify the permutation σ such that $\mathcal{F}$ is still a pairwise suitable family of permutations for $K_n^{1/2}$. To demonstrate this, suppose $v_i \prec_\sigma v_j \prec_\sigma u_{ij}$. Then, we modify σ such that u_{ij} is the immediate predecessor of v_j. It is easy to verify that, for each pair of disjoint edges $e, f \in E(K_n^{1/2})$, if $e \prec_\sigma f$ or $f \prec_\sigma e$ then the same holds in the modified σ too. Similarly, if $u_{ij} \prec_\sigma v_i \prec_\sigma v_j$ then we modify σ such that u_{ij} is the immediate successor of v_i. Hence we can assume the next property also without loss in generality.

Property 2. $v_i \prec_\sigma u_{ij} \prec_\sigma v_j, \forall i, j \in [p], i < j, \ \forall \sigma \in \mathcal{F}$.

These two properties ensure that for any two open intervals (i, j) and (k, l) in C_p if $(i, j) \lhd (k, l)$ then $u_{ij} \prec_\sigma u_{kl}, \forall \sigma \in \mathcal{F}$. In the other case, i.e., when $(i, j) \cap (k, l) \neq \emptyset$, we make the following claim.

Claim 1. *Let $i, j, k, l \in [p]$ such that $(i, j) \cap (k, l) \neq \emptyset$. Then there exist $\sigma_a, \sigma_b \in \mathcal{F}$ such that $u_{ij} \prec_{\sigma_a} u_{kl}$ and $u_{kl} \prec_{\sigma_b} u_{ij}$.*

Since $(i, j) \cap (k, l) \neq \emptyset$, we have $k < j$ and $i < l$. Hence by Property 1, $\forall \sigma \in \mathcal{F}$, $v_k \prec_\sigma v_j$ and $v_i \prec_\sigma v_l$. Now we prove the claim by contradiction. If $u_{ij} \prec_\sigma u_{kl}$ for every $\sigma \in \mathcal{F}$ then, together with the fact that $v_k \prec_\sigma v_j, \forall \sigma \in \mathcal{F}$, we see that no $\sigma \in \mathcal{F}$ can separate the edges $\{v_j, u_{ij}\}$ and $\{v_k, u_{kl}\}$. But this contradicts the fact that $\mathcal{F}$ is a pairwise suitable family of permutations for $K_n^{1/2}$. Similarly if $u_{kl} \prec_\sigma u_{ij}$ for every $\sigma \in \mathcal{F}$ then, together with the fact that $v_i \prec_\sigma v_l, \forall \sigma \in \mathcal{F}$, we see that no $\sigma \in \mathcal{F}$ can separate $\{v_i, u_{ij}\}$ and $\{v_l, u_{kl}\}$. But this too contradicts the pairwise suitability of $\mathcal{F}$. Thus we prove Claim 1.

With these two properties and the claim above, we are ready to prove the following claim.

Claim 2. $|\mathcal{F}| \ge \dim(C_p)$.

For every $\sigma \in \mathcal{F}$, construct a total order L_σ of C_p such that $(i, j) \lhd (k, l) \in L_\sigma \iff u_{ij} \prec_\sigma u_{kl}$. By Property 1 and Property 2, L_σ is a linear extension of $(C_p, \lhd)$. Further, Claim 1 ensures that $\mathcal{R} = \{L_\sigma\}_{\sigma \in \mathcal{F}}$ is a realiser of $(C_p, \lhd)$. Hence $|\mathcal{F}| = |\mathcal{R}| \ge \dim(C_p)$.

Before using the result from Füredi et al. [11] to bound the dimension of the above interval order from below, we assume that $r = |\mathcal{F}| < 2(m+1)/3$, which is safe to do since we have nothing to prove otherwise. This ensures that $n \to \infty$ if and only if $p \to \infty$ and thus $o(1)$ is the class of functions which tend to 0 as either n or p tends to infinity.

From the discussion before the proof, we know that we can bound $\dim(C_p)$ from below by $\lg\lg(p-1) + \frac{1}{2}\lg\lg\lg(p-1) + \frac{1}{2}\lg\frac{\pi}{2} + o(1)$ which in our case is $(m+1-r) + \frac{1}{2}\lg(m+1-r) + \frac{1}{2}\lg\frac{\pi}{2} + o(1)$ since $p = 2^{2^{m+1-r}}$. Hence by Claim 2, $r \geq (m+1-r) + \frac{1}{2}\lg(m+1-r) + \frac{1}{2}\lg\frac{\pi}{2} + o(1)$. Using the fact that $m+1 > \lg\lg(n-1)$, we can conclude that $r \geq \frac{1}{2}\lg\lg(n-1) + \frac{1}{4}\lg\lg\lg(n-1) + \frac{1}{4}\lg\frac{\pi}{4} + o(1)$. The details of computation are left to the reader. □

2.5 Sparsity in graphs with bounded separation dimension

In this section, first we show that graphs with bounded separation dimension cannot be very dense. More precisely we show that an n-vertex graph with separation dimension s is $O((4\lg n)^{s-2})$-degenerate. Next we show that the hypercubes form a sequence of graphs in which separation dimension is a very slowly growing function of the degeneracy.

Proof of Theorem 1.8.

Statement. Every n-vertex graph with separation dimension s, $s \geq 2$, has at most $3(4\lg n)^{s-2}n$ edges.

Proof. Let s be the smallest integer greater than or equal to 2 such that there exists an n-vertex graph G with separation dimension s and $m > 3(4\lg n)^{s-2}n$ edges. We can assume that $s > 2$ since graphs with separation dimension 2 are planar.

Let $\{\sigma_1, \ldots, \sigma_s\}$ be a separating family of permutations for G. After relabelling the vertices if necessary, we can assume that $\sigma_1 = (v_1, \ldots, v_n)$. We define the *length* $l(v_iv_j)$ of an edge v_iv_j in σ_1 as $|i-j|+1$. Setting $l = \lfloor \lg n \rfloor$, for each $k \in [l]$, let B_k be the collection of edges v_iv_j in G such that $\lfloor \lg(l(v_iv_j)) \rfloor = k$. Thus $B_1, \ldots, B_l$ is a partition of $E(G)$. Hence there exists a $k \in [l]$ such that $|B_k| \geq m/\lg n$.

If we choose the set of $\lfloor n/2^k \rfloor$ vertices $H = \{v_{i2^k} : 1 \leq i \leq n/2^k\}$, every edge v_iv_j in B_k is "hit" by a vertex in H, that is, $\exists v_h \in H$ such that $v_i \preceq_{\sigma_1} v_h \preceq_{\sigma_1} v_j$. Hence there exists a vertex $v_h \in H$ such that at least $2^k|B_k|/n$ edges in B_k are hit by v_h. Consider the subgraph G' spanned by the edges in B_k that are hit by v_h. Since the length of an edge in B_k is less than 2^{k+1}, the number of vertices n' in G' is at most 2^{k+2}. One can verify that the number of edges m' in G' which is at least $2^k|B_k|/n$ is more than $3(4\lg n)^{s-3}n'$. Since every pair of disjoint edges in G' is separated in $\{\sigma_2, \ldots, \sigma_s\}$, the separation dimension of G' is at most $s-1$. This contradicts the minimality in the choice of s. □

The final result in the paper is an estimate on the separation dimension of hypercubes. For a positive integer d, the *d-dimensional hypercube* Q_d is the graph with 2^d vertices where each vertex v corresponds to a distinct d-bit binary string $g(v)$ and two vertices $u, v \in V(Q_d)$ are adjacent if and only if $g(u)$ differs from $g(v)$ at exactly one bit position. Let $g_i(v)$ denote the i-th bit from the right in $g(v)$, where $i \in [d]$. The number of ones in $g(v)$ is called the *hamming weight* of v and is denoted by $h(v)$.

Observation 2.7. *Let a, b, c, d be four distinct vertices in the hypercube Q_d with $\{a, b\}, \{c, d\} \in E(Q_d)$ such that $g(a)$ and $g(b)$ differ only in the i-th bit position from the right and $g(c)$ and $g(d)$ differ only in the j-th position from the right. Then there exists some $k \in [d] \setminus \{i, j\}$ such that $g_k(a) = g_k(b) \neq g_k(c) = g_k(d)$.*

Proof. Assume for contradiction that, for every $k \in [d] \setminus \{i, j\}$, $g_k(a) = g_k(b) = g_k(c) = g_k(d)$. If $i = j$ then there can be only 2 distinct binary strings among $\{g(a), g(b), g(c), g(d)\}$. If $i \neq j$, then there can only be 3 distinct binary strings among $\{g(a), g(b), g(c), g(d)\}$ since the i-th and

j-th bit positions from the right cannot simultaneously be $1 - g_i(c)$ and $1 - g_j(a)$ respectively for any of the 4 strings in the set. This contradicts the distinctness of a, b, c, and d. $\square$

Proof of Theorem 1.9.

Statement. For the d-dimensional hypercube Q_d,

$$\pi(K_d^{1/2}) \leq \pi(Q_d) \leq N(d, 3).$$

Proof. The subgraph H of Q_d induced on the vertices with hamming weight 1 and 2 is isomorphic to $K_d^{1/2}$. Hence the lower bound.

Next we show the upper bound by using 3-suitable permutations of the bit positions. Let $\mathcal{E} = \{\sigma_1, \ldots, \sigma_r\}$, $r = N(d, 3)$, be a smallest 3-suitable family of permutations of $[d]$. From $\mathcal{E}$, we construct a family of permutations $\mathcal{F} = \{\tau_1, \ldots, \tau_r\}$ that is pairwise suitable for Q_d. The permutation τ_j is constructed by first permuting the bit positions of all the binary strings according to σ_j and then reading out the vertices in the right-to-left lexicographic order of the bit strings.

In order to show that $\mathcal{F}$ is a pairwise suitable family of permutations for Q_d, consider two disjoint edges $\{a, b\}$, $\{c, d\}$ in Q_d such that $g(a)$ and $g(b)$ differ only in the i-th position from the right and $g(c)$ and $g(d)$ differ only in the j-th position from the right. Then, from Observation 2.7, we know that there exists a $k \in [d] \setminus \{i, j\}$ such that $g_k(a) = g_k(b) \neq g_k(c) = g_k(d)$. Since $\mathcal{E}$ is a 3-suitable family of permutations for $[d]$, there exists a permutation $\sigma_s \in \mathcal{E}$ such that $\{i, j\} \prec_{\sigma_s} k$. Hence, in the right-to-left lexicographic order τ_s, either $\{a, b\} \prec_{\tau_s} \{c, d\}$ or $\{c, d\} \prec_{\tau_s} \{a, b\}$. $\square$

References

[1] Noga Alon, Manu Basavaraju, L. Sunil Chandran, Rogers Mathew, and Deepak Rajendraprasad. Separation dimension of bounded degree graphs. *SIAM Journal on Discrete Mathematics*, 29(1):59–64, 2015.

[2] Manu Basavaraju, L. Sunil Chandran, Martin Charles Golumbic, Rogers Mathew, and Deepak Rajendraprasad. Boxicity and separation dimension. In *Graph-Theoretic Concepts in Computer Science : Proceedings of the 40th International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2014. LNCS*, volume 8747, pages 81–92. Springer International Publishing, 2014.

[3] Manu Basavaraju, L Sunil Chandran, Martin Charles Golumbic, Rogers Mathew, and Deepak Rajendraprasad. Separation dimension of graphs and hypergraphs. *Algorithmica*, 75(1):187–204, 2016.

[4] Manu Basavaraju, L. Sunil Chandran, Rogers Mathew, and Deepak Rajendraprasad. Pairwise suitable family of permutations and boxicity. *arXiv:1212.6756*, 2012.

[5] L. Sunil Chandran, Rogers Mathew, and Naveen Sivadasan. Boxicity of line graphs. *Discrete Mathematics*, 311(21):2359–2367, 2011.

[6] Ben Dushnik. Concerning a certain set of arrangements. *Proceedings of the American Mathematical Society*, 1(6):788–796, 1950.

[7] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[8] Peter C. Fishburn and William T. Trotter. Dimensions of hypergraphs. *Journal of Combinatorial Theory, Series B*, 56(2):278–295, 1992.

[9] Z. Füredi. Scrambling permutations and entropy of hypergraphs. *Random Structures and Algorithms*, 8(2):97–104, 1996.

[10] Z. Füredi. On the Prague dimension of Kneser graphs. *Numbers, information, and complexity*, page 125, 2000.

[11] Z. Füredi, P. Hajnal, V. Rödl, and W.T. Trotter. Interval orders and shift graphs. In *Colloq. Math. Soc. Janos Bolyai*, volume 60, pages 297–313, 1991.

[12] Serkan Hoşten and Walter D Morris. The order dimension of the complete graph. *Discrete Mathematics*, 201(1):133–139, 1999.

[13] H. A. Kierstead. On the order dimension of 1-sets versus k-sets. *Journal of Combinatorial Theory, Series A*, 73(2):219–228, 1996.

[14] Daniel Kleitman. On dedekind's problem: the number of monotone boolean functions. *Proceedings of the American Mathematical Society*, pages 677–682, 1969.

[15] Joel Spencer. Minimal scrambling sets of simple orders. *Acta Mathematica Hungarica*, 22(3-4):349–353, 1972.

[16] W.T. Trotter. New perspectives on interval orders and interval graphs. *London Mathematical Society Lecture Note Series*, 241:237–286, 1997.